BUCKLING MODE CLASSIFICATION OF MEMBERS WITH OPEN THIN-WALLED CROSS-SECTIONS BY USING THE FINITE STRIP METHOD

Research Report

Sándor Ádány

Johns Hopkins University
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1 Initiation of the work

1.1 General

1.2 Uncertainty in the definition of buckling modes of thin-walled members

1.2.1 Definition of buckling modes

1.2.2 Problems, questions with the buckling mode definitions

1.3 Problems in the calculation of critical forces/stresses

1.3.1 Method for defining the critical force for a certain buckling mode

1.3.2 Problems

1.4 Limitations of the calculation methods for buckling analysis

1.4.1 Analytical methods

1.4.2 Finite Element Method (FEM)

1.4.3 Generalized Beam Theory (GBT)

1.4.4 Finite Strip Method (FSM)

1.5 Aims of the work

1.6 General notations

2 Implementation of GBT assumptions into FSM in case of open, single-branch cross-sections

2.1 General

2.2 The effect of GBT strain assumptions on the strips’ local DOFs

2.3 Relationship between the longitudinal DOFs and transverse translational DOFs of main nodal lines

2.4 Relationship between the transverse translational DOFs of main nodal lines and the other transverse DOFs

2.4.1 General

2.4.2 Calculation of the nodal (redundant) moments

2.4.3 Nodes at internal supports

2.4.4 Nodes at external supports

2.4.5 The first and last nodes

3 Implementation of GBT assumptions into FSM in case of open, multi-branch cross-sections

3.1 General

3.2 Relationship between the longitudinal DOFs and transverse translational DOFs of main nodal lines

3.3 Relationship between the transverse translational DOFs of main nodal lines and the other transverse DOFs

3.3.1 General

3.3.2 Element stiffness matrix
1 Initiation of the work

1.1 General

During the last decades an important tendency of the civil engineering industry is the application of more-and-more slender elements, structures. A characteristic appearance of this tendency is the wider and wider application of cold-formed steel members, which is supported not only by the development of the production technology but also by the improvement in design methods, design standards and, in general, computation techniques.

The application of slender elements, however, requires the appropriate handling of buckling phenomena which, probably, are the most important factors in calculating the design capacity. The classical approach to determine the design capacity associated with buckling is to calculate the corresponding linear critical force first (or stress, load coefficient), then to consider the degradation caused by the various kinds of imperfections as well as the possible favorable effect of post-buckling reserve. This second step can most frequently be performed by the introduction of buckling curves which define the relationship between the elastic critical load and the ultimate load in function of the element slenderness.

The modern cold-formed design specifications (Eurocode 3, 2002, NAS 2001, DSM 2004, AS/NZS 1996) basically follow the above-mentioned logic, though sometimes indirectly. What is sure, however, that all the specifications require the correct calculation of the member critical load, since the design load is dependent on it. Evidently, not only the value of the critical load is crucial, but the buckling type, too, to be able to properly consider the effect of imperfections and the possible post-buckling reserves.

In case of cold-formed steel members subjected to compressive axial force and/or bending moment usually three types of basic buckling phenomena are distinguished: local, distortional and global buckling. Each of them has its characteristic post-buckling behavior. It is well-known that local buckling may have significant post-buckling reserve (at least for larger slendernesses where the behavior is primarily elastic). Distortional buckling may have post-buckling reserve, too, but considerably less than for local buckling; while global buckling (such as flexural buckling of a column) has no post-buckling reserve at all, but the real capacity of the member is always less than its elastic critical load. Thus, it is extremely important to clearly classify the various buckling modes in order to get realistic design resistance.

As it will be discussed in detail, the calculation of critical load associated with the various buckling types is still not entirely solved although a number of calculation methods exists. The basic problem is that general methods (such as Finite Element Method), which can handle practically any cross-sections and arbitrary boundary and loading conditions, are not able to directly calculate the critical load for a pure buckling mode, while, on the other hand, the methods which successfully solve the pure mode calculation are not general enough.

The research, summarized in this Report, proposes a new approach to the problem, by developing a method which may enable general numerical methods to directly calculate critical loads for pure buckling modes. As it will be shown, the proposed method can also be applied for the modal decomposition of a general (interacted) buckling mode.
1.2 Uncertainty in the definition of buckling modes of thin-walled members

1.2.1 Definition of buckling modes

As previously mentioned, usually three types of basic buckling phenomena are distinguished for cold-formed steel members (and more generally: in thin-walled members): local, distortional and global buckling. Although there seems to be a consensus on this classification of buckling modes, there is no consensus on the exact meaning of the modes themselves.

Global buckling can be considered as the simplest and clearest case: global buckling is a buckling mode where the member deforms with no deformation in its cross-sectional shape. Thus, the deformations can be characterized by the displacement and torsion of the system line of the member. Depending on the deformations and the type of loading, further sub-classes can be defined such as: flexural buckling, torsional buckling, flexural-torsional buckling and lateral-torsional buckling. It is worth to mention that these modes can analytically be derived, at least for certain types of cross-sections, boundary conditions, etc.

Local buckling is normally defined as the mode which involves plate-like deformations alone, without the translation of the intersection lines of the adjacent plate elements. Another important feature of local buckling is that the associated buckling length is the smallest among the three modes, and typically less than the width of any plate that construct the cross-section.

Distortional buckling seems to be the most problematic mode. As far as the associated buckling length is concerned it is typically in between the lengths of local and global modes, while the transverse deformations involve both plate-like deformations and the translation of one or multiple intersection lines of adjacent plate elements.

In practice it frequently happens that the deformation pattern of the member shows certain features of not only one, but two or three buckling modes. These cases are usually referred as coupled buckling modes.

1.2.2 Problems, questions with the buckling mode definitions

a) Distortional definition by Hancock

Although the notion of distortional buckling is widely used, probably, the only existing definition can be found in the Australian/New Zealand Standard, originated from G. Hancock. It says that distortional buckling is “a mode of buckling involving change in cross-sectional shape, excluding local buckling” with the definition of local buckling as “a mode of buckling involving plate flexure alone without transverse deformation of the line or lines of intersection of adjoining plates”. (AS/NZS, 1996)

To interpret this definition one might conclude that all the modes that are not clearly local or global should be classified as distortional. Although this interpretation is simple, the important consequence is that no coupled modes exist, which, from theoretical aspect, is questionable.
b) What is the line of intersection of two adjacent plate elements? – Example 1

To be able to distinguish between local and distortional buckling, it is crucial to define the meaning of intersection lines. Although the above definitions do not emphasize, they implicitly assume that there are clear intersection lines of the adjoining plate elements of the cross-section. However, this is not always the case.

Figure 1.1 shows a series of cross-sections without and with web stiffeners. The section height is 200 mm, the flange width is 90 mm, the flange lip length is 20 mm, the plate thickness is 2 mm. The web stiffener size parallel with the web is 20 mm, while varies 0 to 8 mm in the perpendicular direction. (Note all dimensions are for the centerline.) The corresponding local/distortional critical forces are shown in Figure 1.2, as a function of the buckling length.

Figure 1.1: C-sections with/without web stiffeners (Example 1)

Figure 1.2: Critical forces for Example 1
Figure 1.2 clearly demonstrates the existence of a minimum point of the curves at approx. 850 mm of buckling length. There is only a little change in the corresponding critical forces, the non-dimensionalized critical force being approx. 0.52-0.53 for all the curves. (Note the non-dimensionalized critical force is calculated as the ratio of the critical force and the squash load.) As far as the buckled cross-section shape is concerned, the associated shapes are identical in nature, all of them having a symmetric pattern with a characteristic vertical edge stiffener movement, as it is expected for a distortional buckling, see Figure 1.3.

Looking at the buckling curves at a smaller half-wave-length, around the “local” minimum, a more considerable change in the critical force can be observed. The associated buckled shapes, at the same time, do not show significant change, as shown in Figure 1.4 where buckled shapes at 170 mm are plotted. Both the above figures indicate that the change in the buckling phenomenon is continuous when the web stiffener size is changing. On the other hand, the buckling mode classification may depend on how the intersection lines are defined.

It is also worth to have a look on the first minimum points of the curves. While the buckling shapes are pretty similar to those presented in Figure 1.4 (a) to (e), the corresponding lengths are gradually increasing as the stiffener becomes stiffer and stiffer. (Note that in case of section (f) this first minimum disappears.) Based on this observation one might conclude that all these shapes should be assigned to the same buckling mode.

However, according to the current practice, the mode shown in Figure 1.4 (a) would certainly be classified as local buckling of the web (although some distortion also takes place), while a similar buckling shape with a web-stiffened section (e.g. Figure 1.4 (e) or (f)) would certainly classified as distortional buckling.
Theoretically, as an extreme case, one could say that all the nodal lines should be considered as effective intersection lines, independently of the relative angle of the adjacent plates. This would lead to the conclusion that all the modes (a) to (e) are distortional. Although this conclusion would be clear theoretically, it definitely contradicts to the current practice.

Another possible case of the buckling mode classification if only those nodal lines are counted as intersection lines the adjacent plates of which have an angle not equal to zero. In this case, the mode for section (a) is local, while distortional for all the other analyzed cross-sections, independently of how small the web stiffener is. It means that there is a discontinuity in the classification which seems to be in contradiction of the seemingly continuous behavior.

It would also be possible to define a threshold value for the relative angle of two adjacent plates, which can define whether an actual nodal line should or should not be consider as intersection line. Depending on the threshold value, some further modes in the above example may fall into the local class. However, the problem of discontinuity in the mode classification is still there.

c) Cantilevers? (Edge stiffeners?) – Example 2

There is a question how to handle edge stiffeners, especially: whether to consider edge-lines of open sections as fictitious intersection lines or not when classifying the buckling mode. The question is similar in nature to the previous one, as the following example shows.

Figure 1.5 shows a series of channel sections with edge stiffeners, however, the angle of the stiffener to the flange changes. As an extreme case, section (a) has a “stiffener” whose angle to the flange is 180 degree, thus, in practice, this is an unstiffened channel section. The section height in all the cases is 200 mm, the thickness is 1 mm, the flange width is 120 mm and the edge stiffener is 10 mm. For section (a) it gives a total flange width of 130 mm, in practice. (All dimensions are for the centerlines.) The corresponding critical forces (as a function of buckling length) are presented in Figure 1.6.

![Figure 1.5: Channel sections with/without edge stiffeners (Example 2)](image-url)
Analyzing the curves of Figure 1.6, it is clear that the sections with very weak stiffeners (or with no stiffener at all) do not show two separate (that is: local and distortional) buckling modes. However, as the angle of the stiffener decreases (that is: the stiffener becomes more and more effective), the section more and more behaves as a lipped channel (or C-section), with both local and distortional modes.
Figure 1.7 presents the buckled cross-section shapes for the modes with a definite end stiffener displacement, at half-wave lengths belonging to the local minimal forces. (For the sections with weak stiffeners this minimum is the only one, while for sections with stronger stiffeners this minimum is the second, distortional one.) Both the critical length and the critical force show an increasing tendency as the stiffener becomes stronger, the length being changed from 300 mm to 850 mm, while the force from 0.025 to 0.094. The buckled shapes, however, all are very similar.

Figure 1.8 presents the buckled cross-section shapes at a length of 170 mm, which, as being less than the web height, may be regarded as local buckling. Moreover, for section with a real edge stiffener this is the length where the first minimum of the buckling curves occur. The figure shows a gradual change from the buckling of the flange of section (a) to the buckling of the web of section (f). Again, the corresponding critical forces are increasing as the stiffener becomes stronger.

All the above figures indicate a gradual and continuous change in the buckling modes, especially the modes with definite stiffener deformations. From this point of view one might conclude that all the modes with the definite translation of the edge-lines should be classified as distortional mode, even if the stiffener practically does not exist. Although this approach is theoretically clear, it has some consequences that are in contradiction with the current practice. Namely:

- Buckling of outstand elements (of a U-section, for example) is traditionally considered as local plate buckling, which “tradition” is reflected also in many design standards.
- Since the web or flange buckling of a C-section always involve the translation of the edge-lines, these modes, usually classified as local modes, cannot be pure local modes any more because of the deformation of the outstand plate elements. More generally: pure local buckling cannot exist in any open section, which, again, in contradiction with the traditional buckling mode classification.

Another possibility is to keep the traditional classification for the sections without edge stiffeners. In this case, however, we must face the same problem of discontinuity as discussed in the previous example.

d) Distortional buckling of I and T sections? – Example 3

In the relevant literature many authors discuss the distortional buckling modes of I and T-sections. (Australian papers, German paper for T-s.) Usually those buckled shapes are referred “distortional” which are seemingly neither pure global nor pure local. From this point of view, the application of “distortional” term for such cases is in accordance with the definition of Hancock.

A characteristic example is presented in Figure 1.9, where a doubly-symmetrical I-section with flanges significantly thicker than the web is shown. (Which is, unquestionably, the usual way of creating welded steel sections.) If the member is subjected to uniform bending, a possible buckled shape for medium buckling length is the one presented in Figure 1.9 (b), which is obviously nor a rigid-body-type deformation neither a plate-buckling-type deformation (which should not include translation of lines of intersections of the plate elements). In other words: the buckling is seemingly nor pure global (namely: lateral-torsional) neither pure local (namely: web buckling). Thus, one might reasonably classify this kind of mode as distortional.
As it will be discussed in the subsequent Sections, the only known general method which is able to calculate distortional buckling separately from the other buckling modes is the Generalized Beam Theory. It is easy to understand, as it will be explained later in detail, that for I and T-sections there are no distortional modes at all according to the GBT way of definition! Thus, the presented buckling mode cannot be distortional in GBT sense, but probably a kind of coupled global-local mode.

![Figure 1.9: “Distortional” buckling of I-sections (Example 3)](image)

1.3 Problems in the calculation of critical forces/stresses

1.3.1 Method for defining the critical force for a certain buckling mode

Since the analytical methods for buckling analysis are not general enough, and, at the same time, there are a number of available numerical methods that are able to handle more general problems, it is usual to perform the buckling analysis by using a numerical method, such as Finite Element Method, Finite Strip Method. (These methods will be discussed in more detail in Section 1.4.)

The commonly used numerical methods, however, are not able to classify the calculated buckling mode or modes. In contrary, the output of a typical FEM or FSM analysis is the value of some of the lowest critical load multipliers (in mathematical sense: the eigenvalues) and the associated deformed shapes (in mathematical sense: the eigenvectors). Depending on the density of the applied finitization and depending on the number of calculated eigenvalues/eigenvectors, this output can mean a huge quantity of data which should somehow be evaluated by the user in order to get the required critical force/stress values belonging to local, distortional and global buckling modes.

Although no method is known which would be able to precisely select the local-distortional-global modes from the output data of a general FEM or FSM software, certain guidance can be given. (The most complete one is in the Commentary of Direct Strength Method, see DSM.) Here is a summary.

- The critical forces and buckled shapes should be calculated and plotted as a function of the buckling half-wave length. This curve may significantly simplify the buckling mode classification.
- As far as buckling shape is concerned the definitions in Section 1.2.1 are basically accepted.
- In case of many practical cross-sections the curve of critical forces has two minimum points at smaller wave-lengths, while after a certain wave-length it tends asymptotically to...
zero. A typical curve is presented in Figure 1.10 for a simple C-section identical to the one shown in Figure 1.1 (a). The buckled cross-section shapes at the two minimum points are presented in Figure 1.11. As it can be seen, the shape that belongs to the lesser half-wave length is a typical local mode, while the shape that belongs to the second minimum is a typical distortional mode. Thus, it is reasonable to classify the two modes as local and distortional, respectively, and it is reasonable to select the corresponding critical forces required to calculate the design strength of the member. (Note this technique implicitly assumes that the member is long enough to be able to develop a buckling shape with similar half-wave length as the ones at minimum points. In many practical cases this assumption is realistic.)

- If the calculated critical force curve has more or less than two minimum points, or if it has two minimum points but the corresponding deformed shapes do not obviously local and distortional, it is not easy to pick up the two appropriate critical forces. In these cases the following rules may be applied:
  - The local mode should be associated with a length less than any of the plate widths of the cross-section.
  - The distortional mode should be associated with a length larger than any of the plate widths of the cross-section.
  - The repetition of the buckling analysis with slightly modified cross-section geometry may help to identify the location of local mode and distortional mode lengths.
  - Application of restraints may exclude the presence of certain modes, which can help to identify the characteristic buckling length and/or critical force for certain modes.
  - The visual analysis of the buckled cross-sectional shape also helps the identification of the buckling modes.

![Figure 1.10: Critical force curve with two distinct minimum points (Example 4)](image-url)
1.3.2 Problems

Here, a list of examples is presented where the determination of critical forces is not evident.

a) Example 5

Let us try to calculate the characteristic critical forces of a simple C-section! The section depth is 200 mm, the flange widths are 50 mm, the edge stiffeners are perpendicular to the flanges with 20 mm of length, while the thickness is 1.5 mm. Note all the dimensions are for the plates center-lines. It is also to mention that these dimensions are by far not extreme, but rather can be considered as typical.

The critical forces are calculated by the CUFSM software (see CUFSM), and are shown in Figure 1.12 in function of the buckling length. At the same time, Figure 1.13 presents some of the buckled cross-sectional shapes, for nine different buckling lengths. These lengths are selected so as to uniformly cover all the length domain of practical interest (more exactly: uniformly in logarithmic sense).

As it is clear from the curve of Figure 1.12, there is a minimum at around 150 mm. From the deformed cross-sectional shapes it is also clear that this minimum point belongs to a local buckling mode, since the buckled shapes corresponding to lengths smaller than this approx. 150 mm do not exhibit translation of the intersection lines.

However, for larger lengths no more minimum point of the buckling curve exists. Although it is more-or-less evident from the curve alone that there is certain change in the behavior around 500 mm of half-wave length, it is not evident at all which length and which critical force should be selected as distortional mode. Looking at the deformed shapes, one might also conclude there is a distortional mode somewhere at 500 mm length, however, it is not sure whether pure distortional mode exists. The dilemma is the following: the modes which have definitely no global mode seem to be coupled local-distortional (see 250 mm and 440 mm), while the modes with a deformed shape typical for distortional buckling definitely exhibit certain rigid-body translation (see 772 mm and 1357 mm). Thus, both the existence of pure distortional mode, both the corresponding value of the critical force is uncertain.
Example 6
Let us analyze a C-section with a small web stiffener! The section basically identical with the one of the previous example, while the height of the triangular web stiffener is 20 mm, the width (measured perpendicularly to the web) is 4 mm.

The critical forces are calculated by the CUFSM software, and are shown in Figure 1.14 in function of the buckling length. Figure 1.15 presents some of the buckled cross-sectional shapes, for the same nine lengths as in case of the previous example.

The buckling curve again has only one minimum point, around 250 mm. Since this length is greater than any of the sections flat plates’ widths, this minimum, although it is the first, probably belongs to not local but distortional mode. Let us see the buckled shapes!

If we consider the intersection lines adjacent to the web stiffener as effective intersection lines (which is a logical assumption for the given dimensions), it is clear from Figure 1.15 that no pure local mode exist for this cross-section. Based on the buckled shapes one would say that the mode at 250 mm (which is the curve minimum) is a coupled mode, with certain contribution of two distortional modes (web stiffener and flange stiffener buckling modes) and probably a local mode (plate buckling of the web). Below this length there seems to be an

Figure 1.12: Critical forces for Example 5

Figure 1.13: Buckled cross-section shapes for Example 5 at various buckling lengths

b) Example 6

Let us analyze a C-section with a small web stiffener! The section basically identical with the one of the previous example, while the height of the triangular web stiffener is 20 mm, the width (measured perpendicularly to the web) is 4 mm.

The critical forces are calculated by the CUFSM software, and are shown in Figure 1.14 in function of the buckling length. Figure 1.15 presents some of the buckled cross-sectional shapes, for the same nine lengths as in case of the previous example.

The buckling curve again has only one minimum point, around 250 mm. Since this length is greater than any of the sections flat plates’ widths, this minimum, although it is the first, probably belongs to not local but distortional mode. Let us see the buckled shapes!

If we consider the intersection lines adjacent to the web stiffener as effective intersection lines (which is a logical assumption for the given dimensions), it is clear from Figure 1.15 that no pure local mode exist for this cross-section. Based on the buckled shapes one would say that the mode at 250 mm (which is the curve minimum) is a coupled mode, with certain contribution of two distortional modes (web stiffener and flange stiffener buckling modes) and probably a local mode (plate buckling of the web). Below this length there seems to be an
interaction of the web plate buckling (local mode) and the web stiffener buckling (distortional mode), while for longer wave-lengths the local mode disappears but the global rigid-body translation occurs. Thus, it is not easy to classify the mode at the curve minimum (250 mm), and, more general, we may conclude that the visual examination of the buckled shapes is not able to classify with certainty any of the modes for smaller and intermediate wave-lengths. (For lengths greater than approx. 4000 mm, there is obviously global flexural buckling.)

![Critical forces for Example 6](image)

Figure 1.14: Critical forces for Example 6

![Buckled cross-section shapes for Example 6 at various buckling lengths](image)

Figure 1.15: Buckled cross-section shapes for Example 6 at various buckling lengths

c) **Example 7**

Example 7 is almost identical with Example 6. The only difference is that the dimension of the web stiffener perpendicularly to the web is now slightly larger, namely: 5 mm.

The critical forces are shown in Figure 1.16 in function of the buckling length, while Figure 1.17 presents some of the buckled cross-sectional shapes for the above-mentioned nine different buckling lengths.

Although neither the cross-section nor the corresponding curve of critical forces are not significantly different from those of the previous example, an important difference still exists. In this case the curve has two minimum points: one at about 300 mm, and another one at
approx. 600 mm. Since both of these lengths are significantly larger than the maximum width of any of the plates, both minimum points should correspond to some distortional mode. This statement is clearly justified by the buckled shapes, too. However, again, it is not easy to decide whether the modes associated with these minimum points are pure modes or not.

Another interesting observation is that although the curve of the critical forces does not show any minimum point at smaller lengths, the buckled shapes suggest the existence of pure local buckling, too. For this specific section the local critical force is probably high enough not to have practical importance, however, one might easily disregard the local buckling in a similar situation.

![Graph](image)

**Figure 1.16: Critical forces for Example 7**

![Buckled Shapes](image)

**Figure 1.17: Buckled cross-section shapes for Example 7 at various buckling lengths**

d) **Example 8**

Example 8 is again only slightly differs from Examples 6 and 7. The web stiffener dimension is further increased here to 6 mm.

The critical forces are shown in Figure 1.18, while Figure 1.19 presents the buckled cross-sectional shapes for the same nine different buckling lengths.
The behavior is principally identical with that of Examples 6 and 7. Here, only one minimum exist, around 640 mm. It is clear from the deformed shapes that the associated mode is an interaction of the distortional (flange stiffener buckling) and the global (flexural). The existence of pure distortional mode is again questionable, while the pure local mode seemingly exists, however, no minimum point is associated with it.

![Figure 1.18: Critical forces for Example 8](image1.png)

**Figure 1.18: Critical forces for Example 8**

![Figure 1.19: Buckled cross-section shapes for Example 8 at various buckling lengths](image2.png)

**Figure 1.19: Buckled cross-section shapes for Example 8 at various buckling lengths**

e) Example 9

Example 9 is again only slightly differs from Examples 6, 7 and 8, its main dimensions being the same. The web stiffener dimension is now 4.5 mm, while the thickness is 0.75 mm.

The critical forces are shown in Figure 1.20, while Figure 1.21 presents the buckled cross-sectional shapes for the same nine different buckling lengths.

The behavior is principally identical with that of Examples 6, 7 and 8. Here, however, there are three minimum points, the first one being a local mode, while the second and third are mainly distortional with some interaction with the local and global modes. It is to be noted that for very short buckling lengths the symmetrical and antimetrical web plate bucklings may occur at practically the same critical force. As an example, Figure 1.21 presents the antimetrical one for the shortest wave-length.
Conclusions from the examples
In general the above examples demonstrate that the determination of the local and distortional critical forces is not obvious even for simple sections with typical dimensions.

- The minimum point(s) of the curve of critical forces not always belongs to a pure mode.
- There are cases when only one minimum point exists.
- There are cases when more than one distortional minimum exist.
- The first minimum point (with the least length) does not necessarily belong to a local minimum.
- Although the visual analysis of the buckled shapes helps the buckling mode classification, it does not provide with a precise method.
1.4 Limitations of the calculation methods for buckling analysis

1.4.1 Analytical methods

Analytical methods are always important and useful, since – although they are usually limited to simpler cases only – they mean a reference for any numerical methods.

For thin-walled members these are the global modes where closed-formed solutions are well-known, especially for buckling of columns. Classical solutions for lateral-torsional buckling exist, too, at least for certain bending moment patterns and for certain cross-section types. (e.g. Timoshenko and Gere, 1961)

As far as local buckling is concerned, analytical solutions are available mainly for single plate elements. In real sections, the plate elements interact with each other, which makes the application of single-plate solutions more complicated, however, they still can be applied for certain practical cases. (e.g. Timoshenko and Gere, 1961)

Until the latest years analytical solutions for distortional buckling have not been available. Recently, however, some researchers proposed closed-formed formulae for distortional buckling, as well. (Hancock, Schafer-Peköz, Camotim-Silvestre, refs??)

1.4.2 Finite Element Method (FEM)

FEM is definitely the most well-known and most popular numerical method in engineering. As in many other engineering problems, it is successfully applied in the buckling load prediction of thin-walled members, too.

a) Advantages of FEM

- FEM can be applied for practically any problems, including various boundary and loading conditions, variable cross-sections, etc.
- FEM softwares are widely available, even in smaller engineering offices.
- There is a kind of tradition of using FEM: researchers as well as design engineers are more and more familiar with both the theoretical background and the practical application of the method.

b) Disadvantages of FEM

- The accurate prediction of critical forces/stresses of thin-walled members requires a relatively large number of finite elements, consequently, a relatively large number of DOFs. In practice the large number of DOFs means the necessity of manipulation of large matrices, vectors, which, again, means significant computational effort. Considering the capacity of today’s computers, however, this significant computational effort does not mean (or means in lesser and lesser extent) important computational time in the buckling analysis of structural members, thus, this traditional disadvantage of the FEM gradually disappears, or have already been disappeared.
- Although the calculation of critical forces and the corresponding buckled shapes is relatively fast, it may be time-consuming to evaluate the results, since it may need to consider a large number (say, a few hundred) of critical loads and buckled shapes to be able to select those few important ones which correspond to the local, distortional and global modes required by the design standards. Moreover, there is no way to simply classify the buckling modes: the selection of the local, distortional and global modes...
therefore necessarily depends on the engineering judgment of the user, based on the
(mainly) visual analysis of the deformation patterns.

- Although there are a number of available FEM softwares, these softwares are typically
general purpose FEM softwares, not containing special tools developed for the calculation
of thin-walled members. At the same time, the correct modeling of members in hand
requires the application of large number of shell finite elements. Without these special
tools, however, it is relatively easy to make mistakes in the modeling, especially in the
boundary conditions and loading, which may have significant influence on the results.

- Some of the design process requires the critical force calculation on not only a single
member, but as a function of the buckling wave length. (This idea is directly included in
the Direct Strength Method, but implicitly included in the Eurocodes and the
Australian/New Zealand Standard.) Although this kind of procedure may help to identify
the buckling modes, by using a general purpose FEM software, it makes the calculation
process much more complicated. Thus, to be able to complete this kind of calculation,
some programming work is definitely necessary for pre- and post-processing, which
highly reduces the effectiveness of FEM.

1.4.3 Generalized Beam Theory (GBT)

GBT is an important method recently developed and used in the analysis of thin-walled
members. (Schardt 1989, Davies and Leach 1994, Davies et al. 1994, Silvestre and Camotim,
2002a, Silvestre and Camotim, 2002b)

a) Advantages of GBT

- Among all the numerical methods GBT works with the least number of DOFs, which
  makes it fast.

- GBT is the only known method that is able to produce pure buckling modes separately
  from the others. This feature makes GBT not only unique but also attractive from design
  point of view, since the required minimal buckling stresses associated with any buckling
  mode can easily be calculated.

b) Disadvantages of GBT

- The method is hard to understand.

- Not widely used, even not widely known. Only a few researchers use it.

- There is no publicly available software working with GBT.

- There are some basic assumptions of the GBT, concerning the strains in the member,
  (namely: transverse membrane strains and shear membrane strains are zero,) the effects of
  which are usually considered to be practically negligible, however, there may be cross-
  sections where these assumptions are not exactly valid.

- GBT can handle prismatic members only.
1.4.4 Finite Strip Method (FSM)

The basic idea of the Finite Strip Method that a finitization and shape functions similar to those of the FEM are used in the transverse direction, however, no finitization is applied in the longitudinal direction. Instead: a specially selected shape function(s) is used, which is able to describe the displacement pattern within the whole length of the member. (For buckling problems of two-hinged members, this longitudinal shape function is a single or multiple sine-waves.) (refs??)

a) Advantages of FSM

- FSM requires much less number of DOFs than FEM, consequently, it is computationally more effective.
- There are a few publicly available FSM softwares, which are developed specially to thin-walled members, therefore, they are easy to use. (see Thin-Wall, CUFSM)
- The available FSM softwares are able to automatically calculate the critical forces and buckled shapes as a function of the buckling length, which simplifies the identification of the buckling modes.
- The theoretical background of FSM is basically similar to that of FEM, thus, people who are familiar with FEM can easily understand FSM, too.

b) Disadvantages of FSM

- FSM much less general than FEM. Its application limits are basically identical with those of GBT. In practice: only prismatic members can be handled.
- An important limitation of FSM that, since it assumes the same longitudinal shape function for all the strips, it cannot provide with buckling modes with different wavelengths in the different strips, which can be critical in certain sections.
- Similarly to FEM, it may be problematic to identify the critical forces associated with global, distortional and local buckling modes (even if the critical forces are automatically calculated in function of the buckling length), since FSM is not able to provide with pure buckling modes, and not able to classify a calculated mode, neither.

1.5 Aims of the work

Based on the problems mentioned in the previous Sections, a number of needs may be identified.

Most of the immediate needs are associated with the requirement of calculating the buckling forces for local, distortional and global modes. Since the critical forces for global modes are relatively easy to calculate for most of the practical cases, the real problem is the identification of local and distortional modes and the calculation of the corresponding buckling forces/stresses.

It is worth to mention that the question is not purely theoretical. The design methods of practically all modern design standards (for cold-formed members) directly require the value of critical stress/force for local and distortional buckling, and it is easy to demonstrate that the design calculation process as well as the resulting design strength/resistance is highly dependent on the proper calculation of the critical stresses.

Thus, the following needs should be mentioned.
1) Clear, unambiguous definitions for the global, distortional and local buckling modes should be given. Among others,
   - the meaning and importance of lines of intersections of plate elements should be clarified,
   - the effect of outstand elements should be clarified,
   - the number of global and distortional modes should be defined.

2) Based on the definitions of pure modes, a method should be developed for the classification of buckling modes which is calculated by means of any numerical method listed or not listed above. The classification method should be able:
   - to identify pure modes,
   - to define the contribution of modes in case of interacted modes.

3) A calculation method should be developed which combines the advantages of the available numerical methods, but ignores their disadvantages in the possible biggest extent. In an ideal case, the method should:
   - be able to calculate pure modes separately from the others as well as to calculate interacted modes,
   - be able to calculate members with irregular restraints,
   - be able to calculate members with holes,
   - be able to calculate members with variable cross-section,
   - be able to calculate buckling modes that cannot be characterized by one single buckling (half sine-wave) length.

4) Since a method which satisfies the above-listed criteria must definitely based on some kind of finitization technique, an appropriate computer tool is also required. This software should:
   - be based on the calculation method as described above,
   - be easy-to-use, such as the available FSM softwares,
   - probably be based on FEM-background, since this is the most well-known and most widely used numerical method.

5) It should be studied how the definitions of the various buckling modes can be extended to cases of less regular members, like
   - members with holes,
   - members with variable cross-sections.

6) The effect of the basic assumptions of GBT (namely: certain membrane strains are assumed to be zero) should be clarified:
   - to determine the cases when it can safely be used,
   - to assess the introduced error if any.
According to the needs listed above, the aims of the work are given here as follows.

1) Construction of pure buckling modes:
   - A method is proposed by which the pure buckling modes can be calculated within the FSM and FEM approach.
   - The implementation of proposed method into FSM will be presented in detail, including the derivation of the necessary formulae and discussion on the applicability.
   - The derived formulae will be implemented into the CUFSM software.

2) Modal decomposition:
   - A method will be proposed to calculate the contribution of the individual modes in a general buckling mode.
   - The modal decomposition method will be implemented in the CUFSM software.

3) Based on the above methods and updated software, the question of buckling mode definition will be discussed.

1.6 General notations

a) Co-ordinate systems

There are basically two co-ordinate systems used throughout this document: a global and a local one, see Figure 1.22. Both are left-handed. The global co-ordinate system is denoted as: X-Y-Z, and it is assumed so that the Y axis would be parallel with the longitudinal axis of the member, otherwise arbitrary. The local system is denoted as x-y-z. The local system is always associated with a plate element of the member so that the x axis would be parallel with the plate element and perpendicular to the member longitudinal axis (in other words: x should lie in the plane of the cross-section), the y axis would be parallel with the member longitudinal axis (in other words: Y and y should coincide), while the z axis is perpendicular to the x-y plane.

Three translations and a rotation is considered as global displacements. The translations are denoted by U-V-W and they are corresponding to the global X-Y-Z co-ordinate axes. (See Figure 1.22) The rotation is denoted by $\theta$ and it means the rotation about the longitudinal axis.

Local displacements are associated with the deformation of a single plate element. Similarly to the global displacements, three translations (u-v-w) and a rotation ($\theta$) is considered. According to the basic feature of the Finite Strip Method the element deformation is expressed in the function of the nodal displacements. The considered nodal displacements are also illustrated in Figure 1.22.

Note, the above-described notations for the co-ordinate systems and global/local displacements are identical with those used in the documentation of the CUFSM software.
b) Node classes

In order to be able to handle a general, multi-branch cross-section with sub-divided plates, we will need to somehow classify the nodes.

Sub nodes (ns) are the nodes that are used to sub-divide the plates. Thus, characteristic features of a sub-node that:

- only two elements are connected to it,
- the two connecting elements lay in the same plain.

All the other nodes are called as main nodes (nm), which, however, can be further divided.

There shall certainly some main nodes to which only one plate element is connected, similarly to the first and last node of an open, single-branch cross-section. These nodes will be referred as external main nodes (nme), or more simply: end nodes. All the other main nodes can be referred as internal main nodes (nmi), or in short: corner nodes.

As it will be shown in the following Chapters, we will try to express all the DOFs of the member in function of the longitudinal displacements of the main nodes (at least for global and distortional buckling modes). As it will be shown below, in case of a multi-branch cross-section the number of independent global and distortional buckling modes are smaller than the number of nodes, even smaller than the number of the main nodes. Thus, it will be necessary to somehow select some main nodes which will be used to determine the other DOFs. Thus, the nodes, in the function of their longitudinal displacements all the other DOFs are expressed, will be called as determining main nodes (nmd), while the other main nodes will sometimes be referred as un-determining main nodes (nmu).
2 Implementation of GBT assumptions into FSM in case of open, single-branch cross-sections

2.1 General

a) Strategy

In this Chapter a method is presented which enables the Finite Strip Method to calculate pure buckling modes similarly to the way as GBT does. The method is based on the observation that GBT is working with smaller number of Degrees of Freedom than FSM, consequently, it is reasonable to assume that FSM includes all the deformation modes that are included in GBT. It also means that the application of appropriate constraints in the FSM may lead to exactly the same solutions that are provided by the GBT, including the same buckling modes.

In practice, the constraints are implemented via the introduction of the GBT assumptions into the FSM, starting with the basic FSM shape functions, then deriving all the necessary formulae. Note the FSM here is identical with the one implemented in the CUFSM software, see the corresponding documentation. (refs)

The application of GBT constraints is also resulted in the reduction of effective Degrees of Freedom. Here, the same way will be followed that normally followed by GBT, namely: the transverse DOFs are expressed in the function of the longitudinal nodal displacements when possible.

b) Notation

An open, single-branch cross-section can conveniently handled by introducing a numbering system shown in Figure 2.1. The total number of nodes (or nodal lines) is \( n \), therefore, the total number of plate elements (or strips) is \((n-1)\). To a general, internal node always two plates are connected, while to the first and last nodes (external nodes) only one single plate element.

Thus, to describe the member, one must define only:

- \( X \) and \( Z \) (global) co-ordinates of all the \( n \) nodes,
- thickness \((t)\) of all the \((n-1)\) plates,
- member length \((a)\),
- material properties.

Note that the width and angle \((b \text{ and } \alpha, \text{ respectively})\) of the plates, which can be easily calculated from the nodal co-ordinates, will also be used, with \( \alpha \) being the angle to the positive \( X \) axis.
c) Simplifying assumptions
For the sake of easier understanding, first, some simplifying assumptions are used, which later will be removed. These assumptions are listed as follows.

- Constant thickness
- Constant, isotropic material
- Strict order of node numbering
- No sub-nodes

2.2 The effect of GBT strain assumptions on the strips’ local DOFs

a) Local $u$-$v$ relationship
The displacements in FSM are expressed as a product of the shape functions and the nodal displacements. For membrane displacements:

$$u(x,y) = \left[ 1 - \frac{x}{b} \right] \left( \frac{x}{b} \right) \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \sin \frac{m \pi y}{a}$$ (2.1)

$$v(x,y) = \left[ 1 - \frac{x}{b} \right] \left( \frac{x}{b} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \cos \frac{m \pi y}{a}$$ (2.2)
At the same time, the two basic assumptions of GBT (regarding membrane strains) are:

\[
\varepsilon_x = \frac{\partial u}{\partial x} = 0 \tag{2.3}
\]

\[
\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = 0 \tag{2.4}
\]

Let us see the effect of the first assumption! Substituting (2.1) into (2.3) we get:

\[
\varepsilon_x = \frac{\partial u}{\partial x} = \frac{-u_1 + u_2}{b} \sin \frac{m\pi y}{a} = 0, \tag{2.5}
\]

and since the sine function is generally not equal to zero:

\[u_1 = u_2,\tag{2.6}\]

that is the \(u\) displacements of the strip’s two nodal lines must be identical. It also means that the corresponding shape function is independent from \(x\) and may be rewritten in a simpler form:

\[u(y) = u \sin \frac{m\pi y}{a}\tag{2.7}\]

Let us consider the second assumption! Substituting Eqs. (2.1) and (2.2) into Eq. (2.4), we get:

\[
\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = u \frac{m\pi}{a} \cos \frac{m\pi y}{a} + \frac{-v_1 + v_2}{b} \cos \frac{m\pi y}{a} = 0 \tag{2.8}
\]

which leads to a relationship between the \(u\) and \(v\) nodal displacements as follows:

\[u = (v_1 - v_2) \frac{a}{b m \pi}. \tag{2.9}\]

By introducing

\[k_m = \frac{m \pi}{a}\tag{2.10}\]

the relationship may be rewritten as:

\[u = (v_1 - v_2) \frac{1}{b k_m}. \tag{2.11}\]

or in a vector form:

\[u = \frac{1}{k_m} \begin{bmatrix} 1/b & -1/b \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \tag{2.12}\]

b) Geometrical interpretation

In order to better understand the physical meaning of the above equations, let us consider a single strip of the member, as shown on Figure 2.2.
According to the assumptions of the FSM: (i) the transverse variation of $u$ and $v$ is linear, (ii) the longitudinal variation of $u$ is sinusoidal, and (iii) the longitudinal variation of $v$ is a cosine function. These assumptions are resulted in a general deformation pattern as illustrated in Figure 2.2.

However, if we consider the strain conditions of GBT, a much restricted deformation pattern must be assumed, as presented in Figure 2.3, which is fully determined by the longitudinal displacements of the two nodal lines.

It may be interesting to underline that the nodal displacements in practice mean the amplitudes of the sine or cosine displacement functions, and, since $u$ is assumed to be a sine, while $v$ is assumed to be a cosine function, they take their maximum values in different cross-sections, see Figure 2.2.

![Figure 2.2: General membrane deformations of a strip in FSM](image1)

![Figure 2.3: Membrane deformations of a strip in GBT](image2)
2.3 Relationship between the longitudinal DOFs and transverse translational DOFs of main nodal lines

a) Relationship between local $u$ and global $V$ nodal displacements

Now, let us consider the $i$-th nodal line of the member with open, single-branch cross-section. The connecting plate elements are the $(i-1)$-th and $(i)$-th, as shown on Figure 2.1. The angles of the plate elements (with respect to the positive x-axis) are: $\alpha^{(i-1)}$ and $\alpha^{(i)}$, respectively.

Applying Eq. (2.12) for both plate elements, we get:

$$u^{(i-1)} = \frac{1}{k_m} \begin{bmatrix} 1/b^{(i-1)} & -1/b^{(i-1)} \end{bmatrix} \begin{bmatrix} v_1^{(i-1)} \\ v_2^{(i-1)} \end{bmatrix}$$  \hfill (2.13)

and

$$u^{(i)} = \frac{1}{k_m} \begin{bmatrix} 1/b^{(i)} & -1/b^{(i)} \end{bmatrix} \begin{bmatrix} v_1^{(i)} \\ v_2^{(i)} \end{bmatrix}$$  \hfill (2.14)

Since the $y$-direction of the local and global co-ordinate systems are parallel, the local $v$ and global $V$ nodal displacements are identical for a given nodal line. Thus:

$$v_1^{(i-1)} = V_{i-1}$$
$$v_2^{(i-1)} = v_1^{(i)} = V_i$$
$$v_2^{(i)} = V_{i+1}$$  \hfill (2.15)

Considering Eq. (2.15), Eqs. (2.13) and (2.14) can be rewritten as:

$$\begin{bmatrix} u^{(i-1)} \\ u^{(i)} \end{bmatrix} = \frac{1}{k_m} \begin{bmatrix} 1/b^{(i-1)} & -1/b^{(i-1)} & 0 \\ 0 & 1/b^{(i)} & -1/b^{(i)} \end{bmatrix} \begin{bmatrix} V_{i-1} \\ V_i \\ V_{i+1} \end{bmatrix}$$  \hfill (2.16)

b) Relationship between local $u$ and global $U,W$ nodal displacements

It is well-known that the relationship between the local $x,y$ and the global $X,Y$ co-ordinates can generally be expressed as:

$$\begin{bmatrix} x \\ z \end{bmatrix} = \begin{bmatrix} \cos \alpha & \sin \alpha \\ -\sin \alpha & \cos \alpha \end{bmatrix} \begin{bmatrix} X \\ Z \end{bmatrix}$$  \hfill (2.17)

Thus, the relationship between the global nodal displacements $(U,V)$ and the local $(u,v)$ displacements of the $(i-1)$-th and $(i)$-th plate elements, respectively:

$$\begin{bmatrix} u^{(i-1)} \\ w_2^{(i-1)} \end{bmatrix} = \begin{bmatrix} \cos \alpha^{(i-1)} & \sin \alpha^{(i-1)} \\ -\sin \alpha^{(i-1)} & \cos \alpha^{(i-1)} \end{bmatrix} \begin{bmatrix} U_i \\ W_i \end{bmatrix}$$  \hfill (2.18)

$$\begin{bmatrix} U_{i-1} \\ V_i \\ W_{i+1} \end{bmatrix}$$  \hfill (2.19)
Considering the first rows of Eqs. (2.18) and (2.19), we get the following expression:

\[
\begin{bmatrix}
  u^{(i)} \\
  u^{(i-1)}
\end{bmatrix} =
\begin{bmatrix}
  \cos \alpha^{(i)} & \sin \alpha^{(i)} \\
  -\sin \alpha^{(i)} & \cos \alpha^{(i)}
\end{bmatrix}
\begin{bmatrix}
  U_i \\
  W_i
\end{bmatrix}
\]  

(2.20)

which is actually defines the relationship between the global \(U,W\) displacements of a node and the local \(u\) displacements of the two adjacent plate elements.

c) Relationship between global \(V\) and global \(U,W\) nodal displacements

Since the left-hand side of Eq. (2.16) and (2.20) are obviously identical, their right-hand sides must be equal, too, which leads to the relationship between the longitudinal \(V\) displacements and the transverse \(U\) and \(W\) displacements, as follows:

\[
\begin{bmatrix}
  U_i \\
  W_i
\end{bmatrix} = \frac{1}{k_m} \begin{bmatrix}
  \cos \alpha^{(i)} & \sin \alpha^{(i)} \\
  \cos \alpha^{(i-1)} & \sin \alpha^{(i-1)}
\end{bmatrix} \begin{bmatrix}
  1/b^{(i-1)} & -1/b^{(i)} \\
  0 & 1/b^{(i)}
\end{bmatrix}
\begin{bmatrix}
  V_{i-1} \\
  V_{i+1}
\end{bmatrix}
\]

(2.21)

\[
\begin{bmatrix}
  U_i \\
  W_i
\end{bmatrix} = \frac{1}{k_m} \begin{bmatrix}
  \cos \alpha^{(i)} & \sin \alpha^{(i)} \\
  \cos \alpha^{(i-1)} & \sin \alpha^{(i-1)}
\end{bmatrix}^{-1} \begin{bmatrix}
  1/b^{(i-1)} & -1/b^{(i)} \\
  0 & 1/b^{(i)}
\end{bmatrix}
\begin{bmatrix}
  V_i \\
  V_{i+1}
\end{bmatrix}
\]

(2.22)

The matrix inversion can be done analytically:

\[
\begin{bmatrix}
  \cos \alpha^{(i-1)} & \sin \alpha^{(i-1)} \\
  \cos \alpha^{(i)} & \sin \alpha^{(i)}
\end{bmatrix}^{-1} = \frac{1}{D_i} \begin{bmatrix}
  \sin \alpha^{(i)} & -\sin \alpha^{(i-1)} \\
  -\cos \alpha^{(i)} & \cos \alpha^{(i-1)}
\end{bmatrix}
\]

(2.23)

where \(D_i\) is the determinant of the matrix:

\[
D_i = \sin \alpha^{(i)} \cos \alpha^{(i-1)} - \sin \alpha^{(i-1)} \cos \alpha^{(i)}
\]

(2.24)

Thus, Eq. (2.21) can be rewritten as:

\[
\begin{bmatrix}
  U_i \\
  W_i
\end{bmatrix} = \frac{1}{k_mD_i} \begin{bmatrix}
  \sin \alpha^{(i)} & -\sin \alpha^{(i-1)} \\
  -\cos \alpha^{(i)} & \cos \alpha^{(i-1)}
\end{bmatrix} \begin{bmatrix}
  1/b^{(i-1)} & -1/b^{(i)} \\
  0 & 1/b^{(i)}
\end{bmatrix}
\begin{bmatrix}
  V_{i-1} \\
  V_{i+1}
\end{bmatrix}
\]

(2.25)

The \(U\) and \(W\) displacements can also be expressed separately, as follows:

\[
U_i = \frac{1}{k_mD_i} \begin{bmatrix}
  \sin \alpha^{(i)} & -\sin \alpha^{(i-1)} \\
  0 & 1/b^{(i)}
\end{bmatrix} \begin{bmatrix}
  1/b^{(i-1)} & -1/b^{(i)} \\
  0 & 1/b^{(i)}
\end{bmatrix}
\begin{bmatrix}
  V_{i-1} \\
  V_{i+1}
\end{bmatrix}
\]

(2.26)
As it can be observed from the above equations, the transverse ($U, W$) displacements of a general, internal node can be calculated as a function of the longitudinal ($V$) displacements of the current plus the adjoining nodes. It means that the transverse displacements of all the internal nodes are defined by the longitudinal displacements of the internal + external nodes. At the same time, it also means that the transverse displacements of the external (first and last) nodes are not determined unambiguously.

It may be interesting to highlight that the $U, V$ and $W$ nodal displacements physically mean the amplitudes of the corresponding displacement functions. In case of the transverse displacements, this displacement function is a sinusoidal function, (one or multiple half-sine wave), which means that the maximum value takes place at the middle cross-section of the member, at least for the single-half-wave case.

At the other hand, the longitudinal displacement function is a cosine function, having its maximum at the member ends (for the single-half-wave case). Thus, Eqs. (2.28) and (2.29) define the relationship between the longitudinal nodal displacements at the member ends and the transverse nodal displacements at the middle of the member.

d) Geometrical limitation

Eqs. (2.25) to (2.29) all contains $D_i$ in the denominator. Thus, to be able to interpret the expressions, the case of $D_i$ being equal to zero must be excluded. Looking at Eq. (2.24), one might observe that $D_i$ is dependent solely on the angles of the plate elements. Moreover, Eq. (2.24) can conveniently be rewritten as:

$$D_i = \sin(\alpha^{(i)} - \alpha^{(i-1)})$$

Thus, to exclude the division by zero, $\alpha^{(i)}$ and $\alpha^{(i-1)}$ must not be equal to each other. (Mathematically, their difference must not be equal to $k\pi$, with $k$ being an integer, however, the only case of practical interest is the $k = 0$ case.) Practically it means that all of the adjoining plate elements must be a definite angle difference so that the above-derived equations can be applied. (Later, it will be shown that nodes not conforming with this condition yet can be used somehow.)
e) \( \mathbf{U} - \mathbf{V} \) and \( \mathbf{W} - \mathbf{V} \) relationship for the cross-section

Based on Eqs. (2.28) and (2.29) the relationship between the displacement vectors can also be expressed, as follows.

\[
\begin{bmatrix}
U_2 \\
U_3 \\
\vdots \\
U_{n-2} \\
U_{n-1}
\end{bmatrix} = \frac{1}{k_m}
\begin{bmatrix}
\sin \alpha^{(2)} & -\sin \alpha^{(2)} & -\sin \alpha^{(1)} \\
D_2 b^{(1)} & D_2 b^{(1)} & D_2 b^{(2)} \\
\sin \alpha^{(3)} & -\sin \alpha^{(3)} & -\sin \alpha^{(2)} \\
D_3 b^{(2)} & D_3 b^{(2)} & D_3 b^{(3)} \\
\sin \alpha^{(2)} & -\sin \alpha^{(2)} & -\sin \alpha^{(1)} \\
D_3 b^{(3)} & D_3 b^{(3)} & D_3 b^{(2)} \\
\end{bmatrix}
\begin{bmatrix}
V_1 \\
V_2 \\
\vdots \\
V_{n-2} \\
V_{n-1}
\end{bmatrix} = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}
\tag{2.31}
\]

\[
\begin{bmatrix}
W_2 \\
W_3 \\
\vdots \\
W_{n-2} \\
W_{n-1}
\end{bmatrix} = -\frac{1}{k_m}
\begin{bmatrix}
\cos \alpha^{(2)} & -\cos \alpha^{(2)} & -\cos \alpha^{(1)} \\
D_2 b^{(1)} & D_2 b^{(1)} & D_2 b^{(2)} \\
\cos \alpha^{(3)} & -\cos \alpha^{(3)} & -\cos \alpha^{(2)} \\
D_3 b^{(2)} & D_3 b^{(2)} & D_3 b^{(3)} \\
\cos \alpha^{(2)} & -\cos \alpha^{(2)} & -\cos \alpha^{(1)} \\
D_3 b^{(3)} & D_3 b^{(3)} & D_3 b^{(2)} \\
\end{bmatrix}
\begin{bmatrix}
V_1 \\
V_2 \\
\vdots \\
V_{n-2} \\
V_{n-1}
\end{bmatrix} = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}
\tag{2.32}
\]

Or:

\[
\mathbf{U} = \frac{1}{k_m} \mathbf{S}_1 \mathbf{V} \tag{2.33}
\]

\[
\mathbf{W} = -\frac{1}{k_m} \mathbf{C}_1 \mathbf{V} \tag{2.34}
\]

where

- \( \mathbf{U} \) and \( \mathbf{W} \) are \((n-2)\)-element vectors with the \( U \) and \( W \) displacements for nodal lines 2 to \((n-1)\),
- \( \mathbf{V} \) is a \( n \)-element vector with all the \( V \)-direction nodal displacements,
- \( \mathbf{S}_1 \) and \( \mathbf{C}_1 \) are \((n-2)\times n\) matrices, containing geometrical data of the member cross-section only.

f) The first and last nodes

Since the first and last nodes are basically the same, here, only the first node will be discussed.

Although the transverse displacements of those nodes to which only one single plate element is attached are not defined by the longitudinal displacements, these \( V \) displacements yet provide certain restraint.

Let us apply Eq. (2.12) for the first element!
\[ u^{(1)} = \frac{1}{k_m} \begin{bmatrix} 1/b^{(1)} & -1/b^{(1)} \end{bmatrix} \begin{bmatrix} v_1^{(1)} \\ v_2^{(1)} \end{bmatrix} \]  

(2.35)

In addition, let us consider the relationship between the global \( V \) and local \( v \):

\[ v_1^{(1)} = V_1 \]
\[ v_2^{(1)} = V_2 \]  

(2.36)

Thus, the local \( u^{(1)} \) can be expressed as follows:

\[ u^{(1)} = \frac{1}{k_m} \begin{bmatrix} 1/b^{(1)} & -1/b^{(1)} \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} \]  

(2.37)

Now, let us make the co-ordinate-transformation of the global transverse displacements of the first node! Unlike in case of a “normal” node, only one equation can be written:

\[ u^{(1)} = \begin{bmatrix} \cos \alpha^{(1)} & \sin \alpha^{(1)} \end{bmatrix} \begin{bmatrix} U_1 \\ W_1 \end{bmatrix} \]  

(2.38)

From the equality of the left-hand sides of Eqs. (2.37) and (2.38), we may write:

\[ \begin{bmatrix} \cos \alpha^{(1)} & \sin \alpha^{(1)} \end{bmatrix} \begin{bmatrix} U_1 \\ W_1 \end{bmatrix} = \frac{1}{k_m} \begin{bmatrix} 1/b^{(1)} & -1/b^{(1)} \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} \]  

(2.39)

Since the above expression contains only one equation but two unknowns, it is clear that the \( U_1, W_1 \) displacements are not fully determined by the \( V_1, V_2 \) displacements. Considering, however, that the physical meaning of the right-hand side of Eq. (2.38), (which is identical with the left-hand side of Eq. (2.39),), is the \( u^{(1)} \)-direction component of the \( U_1, W_1 \) displacements, furthermore considering that this \( u^{(1)} \)-direction component is fully determined by the \( V_1, V_2 \) displacements, (as given in Eq. (2.37),) we may conclude that this is the \( u^{(1)} \)-direction component of the first node’s transverse displacement which is fully determined by the longitudinal displacement, while the \( w^{(1)} \)-direction component (which is perpendicular to the plane of the first plate,) is independent from the \( V \) displacements.

g) Reduction in DOF number

By considering the relationships between the longitudinal and transverse displacement DOFs, it is possible to exactly define the number of DOFs which are determined by the \( n \) pieces of longitudinal displacements.

From Eqs. (2.31) and (2.32) it can be seen that the transverse displacements of all the \((n-2)\) internal nodes are determined by the \( V \) DOFs, which means \(2 \times (n-2)\) DOFs altogether.

Eq. (2.39), at the same time, indicates that one DOF of the first and last nodes is also determined by the longitudinal displacements, which means altogether 2 DOFs.

Thus, the total number of DOFs which is determined by the \( n \) longitudinal displacement DOFs is: \((2 \times n - 2)\).
2.4 Relationship between the transverse translational DOFs of main nodal lines and the other transverse DOFs

2.4.1 General

a) Target
It is important to keep in mind that our main target is to derive relationship between the longitudinal displacements and the other DOFs of the member. In the previous Section the transverse translational displacement DOFs of all the internal nodes, as well as a component of the transverse translational displacement DOFs of the first and last nodes are handled. Thus, the DOFs in question are: (i) the \( n \) pieces of rotational DOFs of all the nodes, and (ii) the \( 2 \times 1 \) translational DOF of the first and the last nodes.

b) The equivalent beam problem
It is important to underline that the previous derivations, presented in Section 2.3, are solely based on

- the two basic strain assumptions of the GBT (that is the membrane transverse and the membrane shear strains are assumed to be zero),
- the shape function assumption of FSM (that is the longitudinal distribution follows sine or cosine shape).

This is why all the above equations contain geometrical data only. However, it is also important to understand that the consequences of those two strain assumptions are fully utilized, in other words, there is no way to express more DOFs in function of the longitudinal ones unless further assumptions are introduced.

As a further assumption, we will apply here the one applied by the GBT. In short: the cross-section displacement must be formed so that the resulting transverse bending forces/stresses should be in equilibrium. It should be underlined that this assumption speaks about not all of the resulting forces/stresses, but only the ones associated with the transverse bending, namely: \( \sigma_x \) stresses and the transverse bending moment.

Consequently, the problem what we have to solve is analogous to a simple bending problem of a multi-span beam. The analogy is illustrated in Figure 2.4 where the cross-section (a), the equivalent beam model (b), and the equivalent beam model after the support movements (c) are shown. The main features of the equivalent beam problem can be summarized as follows:

- The equivalent beam’s global geometry is identical with the cross-section geometry, which means that the nodes of the beam and those of the cross-section are identical.
- All the internal nodes are assumed to be supported by a hinged support, while the end nodes (first and last nodes) are not supported. In other words it means that the first and last beam segments are cantilevers. (However, it should be emphasized that the assumed external supports have not the role of restraining the beam from movement, but rather they just mark those locations of the beam model where the displacements can be prescribed, which displacements usually will not equal to zero, see below.)
- The rigidity of the beam is identical with the transverse rigidity of the member. In practice, it is convenient to take a unit-width portion of the member which leads to a
rigidity equal to $Et^3/[12(1-\nu^2)]$, where $E$ is the Young’s modulus, $\nu$ is the Poisson ratio and $t$ is the plate thickness at the current location. (In case of non-isotropic material, the Young modulus and Poisson ratio for the transverse direction should be taken.)

- For the sake of simplicity, although it is theoretically would not be necessary, the transverse bending rigidity is assumed to be constant over the whole cross-section. Again, this assumption is used solely to make the derivations and expressions simpler. In Section 3.3 the whole derivation will be presented with much less restrictions, including cross-sections with variable thickness and/or material properties.

- The normal rigidity of the beam is assumed to be large enough so that the associated elongation/shortening is negligible. Thus, the normal and shear forces of the equivalent beam model should not be considered.

- No external loading is applied on the equivalent beam, but a kinematic loading expressed by the movement of the supports. It is to mention that these support displacements are exactly the $U, W$ displacements studied in Section 2.3.

c) Solution strategy

The above defined equivalent beam problem requires the solution of a statically indeterminate static system. Basically there are two alternatives: the force method and the displacement method. Theoretically, both methods can be used to any problem. Here, the force method will be presented, according to the solution presented in Silvestre and Camotim (2002a). As it will be shown, the force method could lead to a closed-form solution in case of simpler problems (constant thickness, material, no sub-nodes). However, for more complicated problems the displacement method provides with an advantageous technique, which will also be presented for multi-branch cross-sections.

![Figure 2.4: Cross-section and equivalent multi-span beam](image-url)
2.4.2 Calculation of the nodal (redundant) moments

a) Rotation differences at the nodes from $U, W$

According to the procedure of the force method, first, the beam model should be made
statically determinate by releasing some of its restraints. For beams like our model, usually
hinges are introduced at each internal supports. As a consequence, the unknowns are the
moments (stress resultants) that take place at the internal supports.

Applying the support displacements in the statically determinate model, each beam segment
will displace by rigid-body displacements, while at the supports rotation differences occur.
Let us determine these rotation differences!

The environment of the $i$-th node is shown in Figure 2.5. It is easy to realize that the rotation
difference at the $i$-th node is determined solely by the displacements of the two adjoining
beam segments, which, at the same time, are determined by the transverse displacements of
the nodes ($i-1$), ($i$) and ($i+1$).

The absolute rotation of the two beams can be written as follows:

$$\theta^{(i-1)} = \frac{1}{b^{(i-1)}} \left( w_2^{(i-1)} - w_1^{(i-1)} \right)$$  \hspace{1cm} (2.40)

$$\theta^{(i)} = \frac{1}{b^{(i)}} \left( w_2^{(i)} - w_1^{(i)} \right)$$  \hspace{1cm} (2.41)

where $w$ denotes the local out-of-plane displacements.

Thus, the rotation difference:

$$\Delta \theta_i = \theta^{(i)} - \theta^{(i-1)} = \frac{1}{b^{(i)}} \left( w_2^{(i)} - w_1^{(i)} \right) - \frac{1}{b^{(i-1)}} \left( w_2^{(i-1)} - w_1^{(i-1)} \right)$$  \hspace{1cm} (2.42)

Let us apply the co-ordinate transformation for the local $w$-s, as formulated by Eq. (2.17).

![Figure 2.5: The $i$-th node with the adjoining plates of the statically determinate model](image-url)
Substituting Eqs. (2.43) - (2.46) into Eq. (2.42), we get:

\[ \Delta \theta_i = \frac{1}{b^{(i)}} \left[ \left[ -\sin \alpha^{(i)} \cos \alpha^{(i)} \right] \begin{bmatrix} U_{i+1} \\ W_{i+1} \end{bmatrix} - \left[ -\sin \alpha^{(i)} \cos \alpha^{(i)} \right] \begin{bmatrix} U_i \\ W_i \end{bmatrix} \right] - \frac{1}{b^{(i-1)}} \left[ \left[ -\sin \alpha^{(i-1)} \cos \alpha^{(i-1)} \right] \begin{bmatrix} U_{i} \\ W_{i} \end{bmatrix} - \left[ -\sin \alpha^{(i-1)} \cos \alpha^{(i-1)} \right] \begin{bmatrix} U_{i-1} \\ W_{i-1} \end{bmatrix} \right] \]

\[ \Delta \theta_i = \frac{1}{b^{(i)}} \left[ -\sin \alpha^{(i)} \cos \alpha^{(i)} \right] \begin{bmatrix} U_{i+1} \\ W_{i+1} \end{bmatrix} - \frac{1}{b^{(i)}} \left[ -\sin \alpha^{(i)} \cos \alpha^{(i)} \right] \begin{bmatrix} U_i \\ W_i \end{bmatrix} - \frac{1}{b^{(i-1)}} \left[ -\sin \alpha^{(i-1)} \cos \alpha^{(i-1)} \right] \begin{bmatrix} U_{i} \\ W_{i} \end{bmatrix} + \frac{1}{b^{(i-1)}} \left[ -\sin \alpha^{(i-1)} \cos \alpha^{(i-1)} \right] \begin{bmatrix} U_{i-1} \\ W_{i-1} \end{bmatrix} \]

\[ \Delta \theta_i = \left[ \frac{-\sin \alpha^{(i)} \cos \alpha^{(i)}}{b^{(i)}} \right] \begin{bmatrix} U_{i+1} \\ W_{i+1} \end{bmatrix} - \left[ \frac{-\sin \alpha^{(i)} \cos \alpha^{(i)}}{b^{(i)}} \right] \begin{bmatrix} U_i \\ W_i \end{bmatrix} - \left[ \frac{-\sin \alpha^{(i-1)} \cos \alpha^{(i-1)}}{b^{(i-1)}} \right] \begin{bmatrix} U_{i} \\ W_{i} \end{bmatrix} + \left[ \frac{-\sin \alpha^{(i-1)} \cos \alpha^{(i-1)}}{b^{(i-1)}} \right] \begin{bmatrix} U_{i-1} \\ W_{i-1} \end{bmatrix} \]

Or:

\[ \Delta \theta_i = \left[ \frac{-\sin \alpha^{(i-1)}}{b^{(i-1)}} \right] \left( \frac{\sin \alpha^{(i-1)}}{b^{(i-1)}} + \frac{\sin \alpha^{(i)}}{b^{(i)}} \right) \begin{bmatrix} U_{i-1} \\ U_{i+1} \end{bmatrix} - \left[ \frac{-\cos \alpha^{(i-1)}}{b^{(i-1)}} \right] \left( \frac{\cos \alpha^{(i-1)}}{b^{(i-1)}} + \frac{\cos \alpha^{(i)}}{b^{(i)}} \right) \begin{bmatrix} W_{i-1} \\ W_{i+1} \end{bmatrix} \]

It is worth mentioning that, since the rotation difference at node \( i \) is expressed in function of the transverse \( U,W \) displacements from nodes \( (i-1) \) to \( (i+1) \), and since the transverse displacements of any internal node can be expressed from the longitudinal displacements of the actual and its neighboring nodes, finally the rotation difference at node \( i \) can be expressed in function of five longitudinal displacements: \( V_{i-2} \) to \( V_{i+2} \).
Thus, it is clear that the rotation differences can be expressed for nodes 3 to \((n-2)\), as a function of the longitudinal displacements of all the \(n\) nodes. Based on Eq. (2.50), the relationship can be written as follows:

\[
\begin{bmatrix}
\Delta \theta_3 \\
\Delta \theta_4 \\
\vdots \\
\Delta \theta_i \\
\vdots \\
\Delta \theta_{n-2}
\end{bmatrix}
= 
\begin{bmatrix}
s^{(2)} + s^{(3)} - s^{(3)} & 0 & 0 & 0 & 0 & 0 \\
- s^{(3)} + s^{(4)} - s^{(4)} & 0 & 0 & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & - s^{(i-1)} + s^{(i)} & - s^{(i)} & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & - s^{(n-3)} + s^{(n-2)} & - s^{(n-2)} & - s^{(n-2)}
\end{bmatrix}
\begin{bmatrix}
U_2 \\
U_3 \\
\vdots \\
U_i \\
\vdots \\
U_{n-1}
\end{bmatrix}
\]

(2.51)

with the provisional introduction of the notations:

\[
s^{(i)} = \frac{\sin \alpha^{(i)}}{b^{(i)}}
\]

(2.52)

\[
c^{(i)} = \frac{\cos \alpha^{(i)}}{b^{(i)}}
\]

(2.53)

Eq. (2.51) may be re-written in a shorter form as:

\[
\Delta \theta_L = S_2 U - C_2 W
\]

(2.54)

where

- \(U\) and \(W\) are \((n-2)\)-element vectors with the \(U\) and \(W\) displacements for nodes 2 to \((n-1)\), same as in Eqs. (2.33) and (2.34),
- \(\Delta \theta_L\) is a \((n-4)\) element vector with the relative rotation differences for nodes 3 to \((n-2)\), where the subscript \(L\) indicates that the relative rotations are coming from the loading,
- \(S_2\) and \(C_2\) are \((n-4)\times(n-2)\) matrices, containing geometrical data of the member cross-section only.

Some comments:

- Eq. (2.54) defines the relative rotation differences for nodes 3 to \((n-2)\). This means, however, that all rotation differences are defined, since hinges are introduced only at these nodes.
b) Rotation differences from bending moments

Now, let us calculate the rotation differences from the redundant moments! The problem is illustrated in Figure 2.6 where our equivalent beam model is shown, however, for the sake of better visibility, in an “unfolded” way, all the beam segments being aligned to a straight line. The unknown moments and the associated bending moment diagrams are also shown for the \(i\)-th node and for its neighborhood.

![Figure 2.6: Bending moment diagrams from the redundant moment of the force method](image)

Using the elementary theory of elastic beams, the total rotation difference at the \(i\)-th node can be written as the sum of the rotations of the beam ends adjoining to the \(i\)-th node as follows:

\[
\Delta \theta_i = \frac{m_{i-1}b^{(i-1)}}{6EI} + \frac{m_i b^{(i-1)}}{3EI} + \frac{m_i b^{(i)}}{3EI} + \frac{m_{i+1} b^{(i)}}{6EI}
\]

(2.55)

where the first two terms define the end rotation of the \((i-1)\)-th beam, while the last two terms the end rotation of the \(i\)-th beam, and \(EI\) denotes the beam bending rigidity, as discussed above, which is now assumed to be constant along the full length of the beam.

Eq. (2.55) can be re-written as:

\[
\Delta \theta_i = \frac{1}{6EI} \begin{bmatrix} b^{(i-1)} \\ b^{(i)} \end{bmatrix} \begin{bmatrix} m_{i-1} \\ m_i \\ m_{i+1} \end{bmatrix} + \frac{1}{3EI} \begin{bmatrix} b^{(i-1)} \\ b^{(i)} \end{bmatrix} \begin{bmatrix} m_i \end{bmatrix}
\]

(2.56)

While for the whole beam we may write:
\[
\begin{bmatrix}
\Delta \theta_3 \\
\Delta \theta_4 \\
\vdots \\
\Delta \theta_i \\
\vdots \\
\Delta \theta_{n-2}
\end{bmatrix} = \frac{1}{6EI} 
\begin{bmatrix}
2b^{(2)} + 2b^{(3)} & b^{(3)} & 0 & 0 & 0 \\
b^{(3)} & 2b^{(3)} + 2b^{(4)} & b^{(4)} & 0 & 0 \\
0 & b^{(4)} & \ddots & \ddots & \ddots \\
0 & 0 & \ddots & \ddots & \ddots \\
0 & 0 & 0 & b^{(n-3)} & 2b^{(n-3)} + 2b^{(n-2)} \\
0 & 0 & 0 & 0 & \ddots
\end{bmatrix} 
\begin{bmatrix}
m_3 \\
m_4 \\
\vdots \\
m_i \\
\vdots \\
m_{n-2}
\end{bmatrix}
\] (2.57)

or in a shorter form:
\[
\Delta \theta_R = \frac{1}{6EI} B_1 m
\] (2.58)

where
- \(\Delta \theta_R\) is a \((n-4)\) element vector with the relative rotation differences for nodes 3 to \((n-2)\), where the subscript \(R\) indicates that the rotations are calculated from the redundant moments, (note that this vector is identical in size and in the physical meaning of its elements with the one in Eq. (2.54)),
- \(B_1\) is a \((n-4)\times(n-4)\) tri-diagonal matrix, with geometrical data only,
- \(m\) is a \((n-4)\)-element vector with the unknown nodal moments.

Some comments:
- The above \(m\) vector contains all the nodal moments, for nodes from 3 to \((n-2)\), since at the other nodes there are certainly no moments.
- The fact that \(B_1\) contains geometrical data only is due to the simplifying assumptions used throughout this Section, namely: the cross-section thickness and material is assumed to be constant. In more complicated cases \(B_1\) would necessarily contain information on the rigidity distribution, as well.

c) The unknown nodal bending moments

Once the relative rotation differences are expressed from the loading and from the redundant moments, the unknown (redundant) bending moments can easily be determined, taking into consideration that the nodal rotation differences must ultimately equal to zero. Thus, from Eq. (2.54) and Eq. (2.58) the total rotation difference can be expressed as the sum of right-hand sides of the referred equations.
\[
\Delta \theta = \Delta \theta_L + \Delta \theta_R = 0
\] (2.59)
\[
S_2 U - C_2 W + \frac{1}{6EI} B_1 m = 0
\] (2.60)

And finally the unknown moments can be expressed as:
\[
m = 6EI B_1^{-1} \left( -S_2 U + C_2 W \right)
\] (2.61)

Note that \(B_1\) matrix is certainly invertible since it is a tri-diagonal matrix with positive numbers in its three diagonals, thus, the above expression for \(m\) is a solution for any practical problems.
2.4.3 Nodes at internal supports

a) General

As the redundant moments are already expressed, we can calculate the nodal rotations. The solution is somewhat dependent on the position of the nodes, as it will be shown, thus, first, the nodes at internal supports will be discussed.

Let us consider again the \( i \)-th node of our equivalent beam model. According to the logic of the force method, the absolute rotation can be calculated as the sum of (i) the rotation from the loading (calculated from the rigid-body motions of the statically determinate model) and (ii) the redundant moments (calculated from the elastic deformations of the statically determinate model, again).

b) Nodal rotations from the loading

As far as the rotation from the loading is concerned, the rotations of the adjoining elements are already expressed by Eqs. (2.40) and (2.41). Now, it depends on our choice which of the two adjoining beams we select to calculate the nodal rotation, since the two beam-ends necessarily must have the same rotation (which are, of course, represent the nodal rotation). For now, let us select the following beam, that is the \( i \)-th beam segment. The rotation is expressed by Eq. (2.41), which is repeated here:

\[
\theta^{(i)}(i) = \frac{1}{b^{(i)}} \left( w_2^{(i)} - w_1^{(i)} \right)
\]  

(2.62)

Substituting Eqs. (2.45) and (2.46) into Eq. (2.62):

\[
\theta^{(i)} = \frac{1}{b^{(i)}} \left( -\sin \alpha^{(i)} \cos \alpha^{(i)} \begin{bmatrix} U_{i+1} \\ W_{i+1} \end{bmatrix} - \sin \alpha^{(i)} \cos \alpha^{(i)} \begin{bmatrix} U_i \\ W_i \end{bmatrix} \right)
\]  

(2.63)

which can be re-written as:

\[
\theta^{(i)} = \left[ \frac{\sin \alpha^{(i)}}{b^{(i)}} - \frac{\sin \alpha^{(i)}}{b^{(i)}} \right] \begin{bmatrix} U_{i+1} \\ W_{i+1} \end{bmatrix} - \left[ \frac{\cos \alpha^{(i)}}{b^{(i)}} - \frac{\cos \alpha^{(i)}}{b^{(i)}} \right] \begin{bmatrix} U_i \\ W_i \end{bmatrix}
\]  

(2.64)

For all the internal support nodes we may also write:
with the provisional introduction of the notations:

\[ s^{(i)} = \frac{\sin \alpha^{(i)}}{b^{(i)}} \]  
\[ c^{(i)} = \frac{\cos \alpha^{(i)}}{b^{(i)}} \]  

In a shorter form:

\[ \theta_L = S_3 U - C_3 W \]  

(c) Nodal rotations from the redundant moments

The next step is the calculation of nodal rotations from the redundant moments. Similarly to what we have done before, we will calculate the rotation on the \( i \)-th beam segment. However, this rotation has already been expressed, as the last two terms of Eq. (2.55), as follows:
\[ \theta^{(i)} = \frac{m_i b^{(i)}}{3EI} + \frac{m_{i+1} b^{(i)}}{6EI} \]  

\[(2.69)\]

For all the internal support nodes we may also write:

\[ \begin{bmatrix} \theta^{(3)} \\ \theta^{(4)} \\ \vdots \\ \theta^{(i)} \\ \vdots \\ \theta^{(n-2)} \end{bmatrix} = \begin{bmatrix} 2b^{(3)} & b^{(3)} & 0 & 0 & 0 & 0 \\ 0 & 2b^{(4)} & b^{(4)} & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 2b^{(i)} & b^{(i)} & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 2b^{(n-2)} \end{bmatrix} \begin{bmatrix} m_3 \\ m_4 \\ \vdots \\ m_i \\ \vdots \\ m_{n-2} \end{bmatrix} \]

\[(2.70)\]

or in a shorter form:

\[ \theta_R = \frac{1}{6EI} B_2 m \]  

\[(2.71)\]

where

- \( \theta_R \) is a \((n-4)\) element vector with the absolute rotations for the left-hand side ends of beams 3 to \((n-2)\), where the subscript \( R \) indicates that the rotations are calculated from the redundant moments,
- \( B_2 \) is a \((n-4) \times (n-4)\) two-diagonal matrix, with geometrical data only,
- \( m \) is a \((n-4)\)-element vector with the redundant nodal moments, as already used above.

Note:

- A note similar to what was said for \( B_1 \) can be added here for \( B_2 \). Namely: the fact that \( B_2 \) contains geometrical data only is due to the simplifying assumptions being assumed that the thickness and material is constant in the cross-section.

d) The nodal rotations

Now, the nodal rotations can be calculated as the sum of the rotations from the loading and the redundant moments, as:

\[ \theta = \theta_L + \theta_R \]  

\[(2.72)\]

Substituting Eqs. (2.68) and (2.71) into Eq. (2.72):

\[ \theta = S_3 U - C_3 W + \frac{1}{6EI} B_2 m \]  

\[(2.73)\]

Since the redundant moments have already been expressed in function of the transverse nodal displacements, we may substitute Eq. (2.61) into Eq. (2.73) in order to get the nodal rotations in function of the transverse displacements alone:

\[ \theta = S_3 U - C_3 W + B_2 B_1^{-1} \left( - S_2 U + C_2 W \right) \]  

\[(2.74)\]

Re-arranging the above equation:
\[
\theta = \left( S_3 - B_2 B_1^{-1} S_2 \right) U - \left( C_3 - B_2 B_1^{-1} C_2 \right) W 
\]  
(2.75)

where (repeating here):

- \( U \) and \( W \) are the \((n-2)\)-element vectors with the \( U \) and \( W \) displacements for nodes 2 to \((n-1)\), as previously used,
- \( \theta \) is a \((n-4)\) element vector with the absolute rotations of the nodes 3 to \((n-2)\),
- while the other matrices \((B, C \text{ and } S)\) all contain information on the cross-section geometry only. (Nevertheless, in more complicated cross-sections \( B \) matrices should contain information on the bending stiffness distribution along the cross-section.)

Eq. (2.75) expresses the nodal rotations in function of the transverse displacements. However, these latter ones have already been expressed as a function of the longitudinal displacements. Thus, it is easy now to express the nodal rotations from the nodal longitudinal displacements. To do so, we must simply substitute Eqs. (2.33) and (2.34) into Eq. (2.75), as follows:

\[
\theta = \frac{1}{k_m} \left( S_3 - B_2 B_1^{-1} S_2 \right) S_1 V + \frac{1}{k_m} \left( C_3 - B_2 B_1^{-1} C_2 \right) C_1 V 
\]  
(2.76)

or:

\[
\theta = \frac{1}{k_m} \left[ \left( S_3 - B_2 B_1^{-1} S_2 \right) S_1 + \left( C_3 - B_2 B_1^{-1} C_2 \right) C_1 \right] V 
\]  
(2.77)

with the notations previously defined.

### 2.4.4 Nodes at external supports

The rotation at the external supports, namely: at the second and the \((n-1)\)-th nodes, can be calculated basically the same way as in case of internal nodes, however, with some minor differences.

a) Rotation at the second node

Following the logic of Section 2.4.3, first, let us determine the nodal rotation from the loading! This rotation is identical either with the rotation of the first or the second beam segment. In this case, however, the rotation of the second segment is known only, thus, we should apply Eq. (2.41) or Eq. (2.62), as follows:

\[
\theta_{L}^{(2)} = \frac{1}{b^{(2)}} \left( w_2^{(2)} - w_1^{(2)} \right) 
\]  
(2.78)

Substituting Eqs. (2.45) and (2.46) into Eq.(2.78):

\[
\theta_{L}^{(2)} = \frac{1}{b^{(2)}} \left( \sin \alpha^{(2)} \cos \alpha^{(2)} \left[ U_3 \right] - \sin \alpha^{(2)} \cos \alpha^{(2)} \left[ U_2 \right] \right) 
\]  
(2.79)
which can be re-written as:

\[
\theta_L^{(2)} = \left[ \frac{\sin \alpha^{(2)}}{b^{(2)}} \right] [U_2] - \left[ \frac{\cos \alpha^{(2)}}{b^{(2)}} \right] [W_2]
\]

(2.80)

The next step is the calculation of nodal rotations from the redundant moments. Similarly to what we have done before, we will calculate the nodal rotation as the rotation of the left end of the second beam segment. Applying Eq. (2.69), we may write:

\[
\theta_R^{(2)} = \frac{m_2 b^{(2)}}{3EI} + \frac{m_3 b^{(2)}}{6EI}
\]

(2.81)

Considering, however, that the moment at the second node is zero, the above equation can be simplified as:

\[
\theta_R^{(2)} = \frac{m_3 b^{(2)}}{6EI}
\]

(2.82)

Now, the nodal rotations can be calculated as the sum of the rotations from the loading and the redundant moments, as:

\[
\theta_2 = \theta_L^{(2)} + \theta_R^{(2)}
\]

(2.83)

Substituting Eqs. (2.80) and (2.82) into Eq. (2.83):

\[
\theta_2 = \left[ \frac{\sin \alpha^{(2)}}{b^{(2)}} \right] [U_2] - \left[ \frac{\cos \alpha^{(2)}}{b^{(2)}} \right] [W_2] + \frac{m_3 b^{(2)}}{6EI}
\]

(2.84)

Since the redundant moments have already been expressed in function of the transverse nodal displacements, the absolute rotation of the second node can be expressed in function of the transverse displacements. In practice, we should consider that \( m_3 \) is the first element of the \( m \) vector which is expressed by Eq. (2.61).

Furthermore, by taking into consideration the existing relationship between the longitudinal and the transverse nodal displacements expressed by Eqs. (2.33) and (2.34), it is clear that the nodal rotation of the second node is ultimately the function of the longitudinal displacements. (The details of these two latter steps are not shown here, since it would add little to what already has been said, while would require quite a few new equations with the introduction of new vectors, matrices.)

b) Rotation at the \((n-1)\)-th node

Here, the same steps should be followed as for the second node. First, let us determine the nodal rotation from the loading, which conveniently can be done by expressing the rotation of the right end of the \((n-2)\)-th beam segment:

\[
\theta_L^{(n-2)} = \frac{1}{b^{(n-2)}} \left( w_2^{(n-2)} - w_1^{(n-2)} \right)
\]

(2.85)
Substituting Eqs. (2.45) and (2.46) into Eq.(2.85):

\[
\theta_L^{(n-2)} = \frac{1}{b^{(n-2)}} \left[ \sin \alpha^{(n-2)} \cos \alpha^{(n-2)} \begin{bmatrix} U_{n-1} \\ W_{n-1} \end{bmatrix} - \sin \alpha^{(n-2)} \cos \alpha^{(n-2)} \begin{bmatrix} U_{n-2} \\ W_{n-2} \end{bmatrix} \right] (2.86)
\]

which can be re-written as:

\[
\theta_L^{(n-2)} = \begin{bmatrix} \sin \alpha^{(n-2)} \\ b^{(n-2)} \end{bmatrix} - \begin{bmatrix} \cos \alpha^{(n-2)} \\ b^{(n-2)} \end{bmatrix} \begin{bmatrix} U_{n-2} \\ U_{n-1} \end{bmatrix} - \begin{bmatrix} \cos \alpha^{(n-2)} \\ b^{(n-2)} \end{bmatrix} - \begin{bmatrix} \cos \alpha^{(n-2)} \\ b^{(n-2)} \end{bmatrix} \begin{bmatrix} W_{n-2} \\ W_{n-1} \end{bmatrix} (2.87)
\]

The nodal rotation from the redundant moments is, again, calculated as the rotation of the right end of the (n-2)-th beam segment. We should apply the first two terms of Eq. (2.55), which lead to:

\[
\theta_R^{(n-2)} = \frac{m_{n-2}b^{(n-2)}}{6EI} + \frac{m_{n-1}b^{(n-2)}}{3EI} (2.88)
\]

Considering, however, that the moment at the actual (n-1)-th node is zero, the above equation can be simplified as:

\[
\theta_R^{(n-2)} = \frac{m_{n-2}b^{(n-2)}}{6EI} (2.89)
\]

Now, the nodal rotations can be calculated as the sum of the rotations from the loading and the redundant moments, as:

\[
\theta_{n-1} = \theta_L^{(n-2)} + \theta_R^{(n-2)} (2.90)
\]

Substituting Eqs.(2.87) and (2.89) into Eq. (2.90):

\[
\theta_{n-1} = \begin{bmatrix} \sin \alpha^{(n-2)} \\ b^{(n-2)} \end{bmatrix} - \begin{bmatrix} \cos \alpha^{(n-2)} \\ b^{(n-2)} \end{bmatrix} \begin{bmatrix} U_{n-1} \\ U_{n-2} \end{bmatrix} - \begin{bmatrix} \cos \alpha^{(n-2)} \\ b^{(n-2)} \end{bmatrix} - \begin{bmatrix} \cos \alpha^{(n-2)} \\ b^{(n-2)} \end{bmatrix} \begin{bmatrix} W_{n-2} \\ W_{n-1} \end{bmatrix} + \frac{m_{n-3}b^{(n-2)}}{6EI} (2.91)
\]

Similarly to what has been said for the second node, the rotation of the (n-1)-th node could be expressed in function of the transverse or longitudinal displacements alone, by simply considering that \(m_n\) is the last element of the \(m\) vector, see Eq. (2.61).

### 2.4.5 The first and last nodes

a) Rotation of the first and last nodes

Since the first and last segments of the equivalent beam model, presented in Figure 2.4, are cantilevers, and since there are no external loading on these cantilevers, it is evident that these segments remain straight (in the transverse direction). Consequently, the rotation of any cross-section within the given cantilever is the same. Thus:
\[ \theta_1 = \theta_2 \] 

and 

\[ \theta_n = \theta_{n-1} \]

where \( \theta_2 \) and \( \theta_{n-1} \) are expressed by Eqs. (2.84) and (2.91), respectively.

b) Translation of the first and last nodes

The only DOFs that are not determined yet are the translational DOFs of the first and last nodes. As it was explained in Section 2.2 under the point f), this is the perpendicular (to the corresponding plate) component of the first and last nodal displacement which is not determined by the longitudinal displacements. This perpendicular component, however, is obviously determined by the equivalent beam model.

Utilizing the simplifications due to the assumption of small displacements (which is implicitly assumed throughout this document,) the translation of the first node can be written as follows:

\[ w_1^{(1)} = w_2^{(1)} + \theta^{(1)} b^{(1)} \]  

Applying Eq. (2.46), we may write:

\[ w_2^{(1)} = \begin{bmatrix} -\sin\alpha^{(1)} & \cos\alpha^{(1)} \end{bmatrix} \begin{bmatrix} U_2 \\ W_2 \end{bmatrix}, \]  

(2.95)

At the same time, \( \theta^{(1)} \) is equal to \( \theta_1 \) or \( \theta_2 \), expressed by Eq. (2.84), thus, the undetermined component of the translation of first node is expressed in function of the transverse displacements of the other nodes, or, in function of the longitudinal displacements.

As far as the last node is concerned, similar equations may be written:

\[ w_2^{(n-1)} = w_1^{(n-1)} + \theta^{(n-1)} b^{(n-1)} \]  

(2.96)

where

\[ w_1^{(n-1)} = \begin{bmatrix} -\sin\alpha^{(n-1)} & \cos\alpha^{(n-1)} \end{bmatrix} \begin{bmatrix} U_{n-1} \\ W_{n-1} \end{bmatrix}, \]  

(2.97)

and \( \theta^{(n-1)} \) is equal to \( \theta_n \) or \( \theta_{n-1} \), expressed by Eq. (2.91).
3 Implementation of GBT assumptions into FSM in case of open, multi-branch cross-sections

3.1 General
The description of an open, multi-branch cross-section requires slightly more data than a single-branch section. The necessary additional information is how the elements are connected to the nodes (and each other), which is defined by the appropriate ordering of the nodes and elements for single-branch sections. Thus, to describe a multi-branch cross-section, one should define:

- $X$ and $Z$ (global) co-ordinates of all the $n$ nodes,
- start and end-nodes for all the $(n-1)$ plates,
- thickness ($t$) of all the $(n-1)$ plates,
- member length ($a$),
- material properties.

Again, the width and angle ($b$ and $\alpha$, respectively) of the plates can be easily calculated from the nodal co-ordinates.

Unlike in the previous Chapter, here two adjoining plate elements do not need to have a distinct angle difference. In other words, a plate, which is physically one single plate, can be sub-divided into smaller plates laying in the same plane. (The application of this sub-division is necessary to properly calculate local buckling.)

3.2 Relationship between the longitudinal DOFs and transverse translational DOFs of main nodal lines

a) Relationship between local $u$ and global $V$ nodal displacements
Unlike in a single-branch section, in a multi-branch cross-section there is at least one node to which more than two plates are connected. As it will be shown below, this kind of nodes make the main difference in deriving the relationships between the various DOFs, thus, we will first concentrate on a node like this.

Let us consider the $i$-th main node, with $m_i$ adjoining plates, as it is presented in Figure 3.1. For the sake of simplicity, let us number the adjacent main nodes as 1, 2, 3, ..., $j$, ..., $m$. The elements are denoted by their start and end nodes, like $i.1$, $i.2$, ..., $i.j$, ..., $i.m$, the first and second index being the start and end node, respectively. It should be underlined here that there may be sub-nodes within any element, however, these sub-nodes will be handled later. For this reason the Figure shows only the main nodes.
Similarly to what has been done for single-branch cross-sections, let us apply Eq. (2.12) for all the plate elements that are connected to the $i$-th node. Doing so we get $m$ equations of the following kind:

$$u^{(j)} = \frac{1}{k_m} \begin{bmatrix} 1/b^{(j)} & -1/b^{(j)} & v_1^{(j)} & v_2^{(j)} \end{bmatrix} \begin{bmatrix} v_1 \v_2 \end{bmatrix} j = 1 \ldots m \quad (3.1)$$

Considering the relationship of local $v$ and global $V$ displacements, one might write:

$$v_1^{(j)} = V_i \quad j = 1 \ldots m \quad (3.2)$$
$$v_2^{(j)} = V_j \quad j = 1 \ldots m$$

Substituting Eq. (3.2) into Eqs. (3.1):

$$\begin{bmatrix} u^{(i,1)} \\ u^{(i,2)} \\ \vdots \\ u^{(i,j)} \\ \vdots \\ u^{(i,m)} \end{bmatrix} = \frac{1}{k_m} \begin{bmatrix} 1/b^{(i,1)} & -1/b^{(i,1)} & 0 & 0 & 0 & 0 & 0 \\ 1/b^{(i,2)} & 0 & -1/b^{(i,2)} & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1/b^{(i,j)} & 0 & 0 & 0 & -1/b^{(i,j)} & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1/b^{(i,m)} & 0 & 0 & 0 & 0 & 0 & -1/b^{(i,m)} \end{bmatrix} \begin{bmatrix} V_i \\ V_1 \\ V_2 \\ \vdots \\ V_j \\ \vdots \\ V_m \end{bmatrix} \quad (3.3)$$

Figure 3.1: A general main node of a multi-branch cross-section with the adjacent elements and main nodes
b) Relationship between local $u$ and global $U,W$ nodal displacements

Similarly to what has been done for single-branch cross-sections, the usual transformation matrix as defined by Eq. (2.17) can be applied for the $U_i, W_i$ displacements, however, now, it should be applied $m$ times, according to the number of connecting plate elements. Thus, we have $m$ equations of the following kind:

$$
\begin{bmatrix}
    u^{(i,j)} \\
    w^{(i,j)}
\end{bmatrix} =
\begin{bmatrix}
    \cos \alpha^{(i,j)} & \sin \alpha^{(i,j)} \\
    -\sin \alpha^{(i,j)} & \cos \alpha^{(i,j)}
\end{bmatrix}
\begin{bmatrix}
    U_i \\
    W_i
\end{bmatrix}
\quad j = 1 \ldots m
$$

(3.4)

Considering the first row of Eq. (3.4), we may write:

$$
u^{(i,j)} = \begin{bmatrix}
    \cos \alpha^{(i,j)} & \sin \alpha^{(i,j)}
\end{bmatrix}
\begin{bmatrix}
    U_i \\
    W_i
\end{bmatrix}
\quad j = 1 \ldots m
$$

(3.5)

or, in a more detailed way:

$$
\begin{bmatrix}
    u^{(i,1)} \\
    u^{(i,2)} \\
    \vdots \\
    u^{(i,j)} \\
    \vdots \\
    u^{(i,m)}
\end{bmatrix} =
\begin{bmatrix}
    \cos \alpha^{(i,1)} & \sin \alpha^{(i,1)} \\
    \cos \alpha^{(i,2)} & \sin \alpha^{(i,2)} \\
    \vdots & \vdots \\
    \cos \alpha^{(i,j)} & \sin \alpha^{(i,j)} \\
    \vdots & \vdots \\
    \cos \alpha^{(i,m)} & \sin \alpha^{(i,m)}
\end{bmatrix}
\begin{bmatrix}
    U_i \\
    W_i
\end{bmatrix}
$$

(3.6)

which is actually defines the relationship between the global $U,W$ displacements of a node and the local $u$ displacements of the adjacent plate elements.

c) Relationship between global $V$ and global $U,W$ nodal displacements

As it is shown in Section 2.2, the transverse displacements of an internal node are fully determined by the longitudinal displacements of the adjoining two plate elements (or strips). In an open, single-branch section there are only two plate elements connected to any internal nodes, thus, the transverse displacements of all the internal nodes are exactly determined by the longitudinal displacements of the nodes.

In an open, multi-branch cross-section, however, there is at least one node to which more than two plates are connected. If we want to follow the logic of the derivations applied for single-branch sections, we should assume that the longitudinal displacements of all the nodes are known. From Eq. (3.3) it is obvious that these longitudinal displacements unambiguously determine the local $u$ displacements of all the plate elements.

Looking at Eq. (3.6), it is easy to understand that the transverse displacement of a node with multiple connecting plates is over-determined, since there are only two unknowns (namely: $U_i$ and $W_i$) but $m \quad (m>2)$ equations. In other words, any two of the adjacent plate elements (with a distinct angle difference) would unambiguously determine the $U_i, W_i$ displacements, consequently, the rest of the equations are redundant.

If we yet want to satisfy Eq. (3.6), we cannot have arbitrary $u$-s, therefore, we cannot have arbitrary $V$ longitudinal displacements. Mathematically, the problem can be handled as follows.
Since the left-hand side of Eq. (3.3) and (3.6) are identical, their right-hand sides must be equal, too, which leads to the following equation:

\[
\begin{bmatrix}
\cos \alpha^{(i,1)} & \sin \alpha^{(i,1)} \\
\cos \alpha^{(i,2)} & \sin \alpha^{(i,2)} \\
\vdots & \vdots \\
\cos \alpha^{(i,j)} & \sin \alpha^{(i,j)} \\
\vdots & \vdots \\
\cos \alpha^{(i,m)} & \sin \alpha^{(i,m)}
\end{bmatrix}
\begin{bmatrix}
U_i \\
W_i
\end{bmatrix}
= \frac{1}{k_m}
\begin{bmatrix}
\frac{1}{b^{(i,1)}/b^{(i,1)}} - \frac{1}{b^{(i,1)}/b^{(i,1)}} & 0 & 0 & 0 & 0 \\
\frac{1}{b^{(i,2)}/b^{(i,2)}} & 0 & -\frac{1}{b^{(i,2)/b^{(i,2)}}} & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
\frac{1}{b^{(i,j)}/b^{(i,j)}} & 0 & 0 & \cdots & \frac{1}{b^{(i,j)/b^{(i,j)}}} & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
\frac{1}{b^{(i,m)/b^{(i,m)}}} & 0 & 0 & \cdots & \frac{1}{b^{(i,m)/b^{(i,m)}}} & 0
\end{bmatrix}
\begin{bmatrix}
V_i \\
V_1 \\
V_2 \\
\vdots \\
V_j \\
V_m
\end{bmatrix}
\] (3.7)

As it is discussed above, any two equations would determine the unknown \(U_i, W_i\) displacements (provided the corresponding angles are different), thus, let us select the first two equations to calculate the \(U_i, W_i\) transverse displacements, while the rest of the equations must be used to set up a restraint for the \(V\) longitudinal displacements. Partitioning Eq. (3.7) in this way, we may write:

\[
\begin{bmatrix}
\cos \alpha^{(i,1)} & \sin \alpha^{(i,1)} \\
\cos \alpha^{(i,2)} & \sin \alpha^{(i,2)} \\
\vdots & \vdots \\
\cos \alpha^{(i,j)} & \sin \alpha^{(i,j)} \\
\vdots & \vdots \\
\cos \alpha^{(i,m)} & \sin \alpha^{(i,m)}
\end{bmatrix}
\begin{bmatrix}
U_i \\
W_i
\end{bmatrix}
= \frac{1}{k_m}
\begin{bmatrix}
\frac{1}{b^{(i,1)/b^{(i,1)}}} - \frac{1}{b^{(i,1)/b^{(i,1)}}} & 0 & 0 & 0 & 0 \\
\frac{1}{b^{(i,2)/b^{(i,2)}}} & 0 & -\frac{1}{b^{(i,2)/b^{(i,2)}}} & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
\frac{1}{b^{(i,j)/b^{(i,j)}}} & 0 & 0 & \cdots & \frac{1}{b^{(i,j)/b^{(i,j)}}} & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
\frac{1}{b^{(i,m)/b^{(i,m)}}} & 0 & 0 & \cdots & \frac{1}{b^{(i,m)/b^{(i,m)}}} & 0
\end{bmatrix}
\begin{bmatrix}
V_i \\
V_1 \\
V_2 \\
\vdots \\
V_j \\
V_m
\end{bmatrix}
\] (3.8)

or, by introducing some simplifying provisional notations:

\[
\begin{bmatrix}
A_1 \\
A_2
\end{bmatrix}
\begin{bmatrix}
D_T
\end{bmatrix}
= \frac{1}{k_m}
\begin{bmatrix}
B_{11} & B_{12} \\
B_{21} & B_{22}
\end{bmatrix}
\begin{bmatrix}
D_{L1} \\
D_{L2}
\end{bmatrix}
\] (3.9)

The first and second equations of Eq. (3.9) can be written as:

\[
A_1D_T = \frac{1}{k_m}(B_{11}D_{L1} + B_{12}D_{L2})
\] (3.10)
Considering that \( B_{12} \) is a matrix with zero elements only, the \( D_T \) transverse displacements can be expressed from Eq. (3.10) basically the same way as it has been done for single-branch sections, in function of three longitudinal displacements (\( D_{L1} \)): \[
A_2 D_T = \frac{1}{k_m} \left( B_{21} D_{L1} + B_{22} D_{L2} \right) \tag{3.11}
\]

At the same time, Eq. (3.11) can be used to express the longitudinal displacements of the other nodes (\( D_{L2} \)): \[
D_{L2} = B_{22}^{-1} \left( k_m A_2 D_T - B_{21} D_{L1} \right) \tag{3.12}
\]

where the matrix inversion can always be performed \( B_{22} \) being a diagonal matrix with definitely non-zero diagonal elements. Furthermore, let us substitute Eq. (3.12) into Eq. (3.13).

\[
D_{L2} = B_{22}^{-1} \left( A_2 A_1^{-1} B_{11} D_{L1} - B_{21} D_{L1} \right) \tag{3.14}
\]

\[
D_{L2} = B_{22}^{-1} A_2 A_1^{-1} B_{11} D_{L1} - B_{22}^{-1} B_{21} D_{L1} \tag{3.15}
\]

or \[
D_{L2} = \left( B_{22}^{-1} A_2 A_1^{-1} B_{11} - B_{22}^{-1} B_{21} \right) D_{L1} \tag{3.16}
\]

Thus, Eq. (3.16) defines the relationship between the longitudinal displacements of nodes associated with a given node. If there are multiple nodes with more than two adjacent plates, similar equations should be applied.

d) Summary

Summarizing the above considerations the procedure is as follows:

- First the main nodes of the cross-section should be defined.
- For each internal main node to which more than two plates are attached (node \( i \)) two plates should be selected that are not lying in the same plain. The longitudinal displacements of the main nodes at the far end of these two plates (e.g. nodes 1 and 2) will be used to define
  - the transverse displacement of the given \( i \)-th main node,
  - the longitudinal displacements of the main nodes of all the other adjoining plates (e.g. nodes 3 to \( m \)).
- The transverse displacement of the \( i \)-th node can be calculated by using Eq. (3.12).
- The longitudinal displacements of nodes 3 to \( m \) can be calculated by using Eq. (3.16).
- By applying the above steps for all the internal main nodes, the relationship between the determining main nodes and the transverse displacements of all the main nodes can be established.
e) **Some comments:**

The $A$ and $B$ sub-matrices, which determine the condition for the longitudinal nodal displacements, contains only information for cross-sectional geometry, namely: plate widths and angles. Thus, they are independent of the member length.

The above equations can also be applied for nodes with two connecting elements, too. In this case some terms will disappear which will be resulted in basically the same equations as the ones derived for single-branch cases.

It is to be noted that the inverse matrix of $A_1$ can be easily calculated analytically, by using Eqs. (2.23) and (2.24). The inverse matrix exists only if the selected two plates are not parallel. However, in case of a node with more than two adjoining plates there are always two non-parallel ones, therefore, the above procedure can always be performed by the appropriate partitioning of the problem.

It may be interesting to mention that the term $B_{22}^{-1}B_{21}$ is easy to calculate due to the special structure of the involved $B$ sub-matrices:

$$
B_{22}^{-1}B_{21} = \begin{bmatrix}
-1 & 0 & 0 \\
-1 & 0 & 0 \\
-1 & 0 & 0 \\
\vdots & \vdots & \vdots \\
-1 & 0 & 0 
\end{bmatrix}
$$

which is a matrix with $(m-3)$ rows and 3 columns, all the elements of the first column being equal to (-1) while all the other elements of the matrix being equal to zero.

f) **End nodes**

Since the end nodes of a multi-branch cross-section do not substantially differ from the first and last nodes of a single-branch cross-section, everything that was said for these latter nodes is valid for any end nodes of any sections. Thus, Eqs. (2.35) to (2.39) could be applied. Of course, the conclusion would be the same, too, that is the component of the transverse displacement of any end nodes which is perpendicular to the corresponding plate element is independent from the longitudinal displacement DOFs, while the parallel component is fully determined by them.

g) **Sub-nodes**

Since one of the basic assumptions states that the distribution of the longitudinal displacement within a flat plate element is linear, the longitudinal displacements of all the sub-nodes are automatically determined by the longitudinal displacements of the main nodes. In practice, linear interpolation should be used.
3.3 Relationship between the transverse translational DOFs of main nodal lines and the other transverse DOFs

3.3.1 General

a) Target

Similarly to Section 2.4, the main target of this Section is to derive relationship between the longitudinal displacements and the other DOFs of the member. In the previous Section the transverse translational displacement DOFs of all the internal main nodes, as well as a component of the transverse translational displacement DOFs of all the external main nodes are covered. Thus, the DOFs in question are:

- the rotational DOFs of all the main nodes,
- the translational DOFs of the external main nodes (one/node),
- the translational and rotational DOFs of all the sub-nodes (three/node).

b) The equivalent beam problem

Similarly to what we have done for single-branch cross-sections, we have to introduce further assumptions in order to be able to express the rest of DOFs in function of the longitudinal displacements of the determining main nodes. The applied assumption is basically the same: the cross-section displacement must be formed so that the resulting transverse bending forces/stresses should be in equilibrium. (Again: transverse alone!)

As we have done in Section 2.4, an equivalent bending problem can be defined, basically in the same way. However, there are small differences due to the somewhat less restrictive assumptions, thus, the features of the equivalent beam model are listed again as follows:

- The equivalent beam’s global geometry is identical with the cross-section geometry, which means that the nodes of the beam and those of the cross-section are identical.
- All the internal main nodes are assumed to be supported by a hinged support, while the external main nodes are not supported. (Of course, sub-nodes neither.)
- The rigidity of the beam is identical with the transverse rigidity of the member. In practice, it is convenient to take a unit-width portion of the member which leads to a rigidity equal to \( E t^3 /[12(1-\nu^2)] \), where \( E \) is the Young’s modulus, \( \nu \) is the Poisson ratio and \( t \) is the plate thickness at the current location. (In case of orthotropic material, the Young modulus and Poisson ratio for the transverse direction should be taken.) It is to be underlined that \( t, E \) and \( \nu \) must be constant, but within one element only, thus, for example, variable thickness is possible to handle even for one flat plate element by subdividing it with some sub-nodes.
- The normal rigidity of the beam is assumed to be large enough so that the associated elongation/shortening is negligible. Thus, the normal and shear forces of the equivalent beam model should not be considered.
- No external loading is applied on the equivalent beam, but a kinematic loading, expressed by the movement of the supports. These support displacements are exactly the transverse \( U, W \) displacements analyzed in the previous Section.
c) **Solution strategy**

As it was mentioned earlier, there are basically two methods to solve our problem: the force method and the displacement method. The two methods lead to the same results. In Chapter 2 the force method was presented while here the displacement method is used. The application of displacement method requires the following main steps:

- creation of element stiffness matrices for all the beam (plate) segments,
- assembling global stiffness matrix,
- expressing unknown DOFs in function of the known ones.

### 3.3.2 Element stiffness matrix

According to the logic of a displacement method, first, we shall construct the element stiffness matrix of the equivalent beam element in the local co-ordinate system.

#### a) Derivation based on the beam theory

Let us consider a single beam element, as shown in Figure 1.22. The assumed order of the DOFs is: $u_1, w_1, \theta_1, u_2, w_2, \theta_2$.

![Figure 3.2: DOFs of an equivalent beam element](image)

It is well-known that the element stiffness matrix can conveniently be written as follows:

$$
\mathbf{k}_{i,\text{bar}} = \begin{bmatrix}
\frac{EA}{b} & 0 & \frac{12EI}{b^3} & 0 & -\frac{12EI}{b^3} \\
0 & \frac{6EI}{b^3} & \frac{4EI}{b^2} & 0 & -\frac{6EI}{b^3} \\
\frac{12EI}{b^3} & \frac{4EI}{b^2} & \frac{EA}{b} & 0 & -\frac{4EI}{b^2} \\
0 & 0 & \frac{EA}{b} & \frac{4EI}{b^2} & 0 \\
-\frac{12EI}{b^3} & -\frac{6EI}{b^3} & -\frac{4EI}{b^2} & 0 & \frac{12EI}{b^3}
\end{bmatrix} \quad \text{symm.}
$$

(3.18)

where $EA$ and $EI$ are the normal and bending stiffness of the beam, respectively. (Note the consideration of the beam’s longitudinal displacement and the corresponding normal rigidity is included here for the sake of completeness. In practice, to the normal rigidity a high value may be assigned to exclude the beam longitudinal deformation.)
Considering that the beams of our problem are “equivalent”, since they represent plate-like elements, the above stiffness terms should be formulated as (assuming isotropic material):

\[
EA = \frac{Eat}{1 - \nu^2}
\]  
\( (3.19) \)

\[
EI = \frac{Eat^3}{12(1 - \nu^2)}
\]  
\( (3.20) \)

In case of orthotropic plates, slightly modified formulae should be applied, as follows:

\[
EA = E_1A = \frac{E_x at}{1 - \nu_x \nu_y}
\]  
\( (3.21) \)

\[
EI = E1 = \frac{E_x at^3}{12(1 - \nu_x \nu_y)}
\]  
\( (3.22) \)

Note that the longitudinal axes of the equivalent beams always coincide with the local \( x \)-axes of the plates.

Moreover, we can further simplify the expressions for the stiffness terms by introducing the notations of the CUFSM documentation:

\[
E_1 = \frac{E_x}{1 - \nu_x \nu_y}
\]  
\( (3.23) \)

\[
D_x = \frac{E_x t^3}{12(1 - \nu_x \nu_y)}
\]  
\( (3.24) \)

Thus, the beams’ normal and bending stiffnesses can be written as:

\[
EA = E_1 at
\]  
\( (3.25) \)

\[
EI = D_x a
\]  
\( (3.26) \)

while the element stiffness matrix for the equivalent beam problem of an orthotropic plate structure:

\[
k_{i,\text{plate}} = \begin{bmatrix}
\frac{E_1 at}{b} & 0 & \frac{12 D_x a}{b^3} & \frac{4 D_x a}{b} \\
0 & \frac{6 D_x a}{b^2} & 0 & \frac{12 D_x a}{b^3} \\
-\frac{E_1 at}{b} & 0 & 0 & \frac{E_1 at}{b} \\
0 & -\frac{12 D_x a}{b^3} & -\frac{6 D_x a}{b^2} & 0 \\
0 & \frac{6 D_x a}{b^2} & \frac{2 D_x a}{b} & 0 \\
0 & \frac{6 D_x a}{b^2} & \frac{2 D_x a}{b} & 0
\end{bmatrix}_{\text{symm.}}
\]  
\( (3.27) \)
b) Derivation based on the Finite Strip Method

It is interesting to mention that the same result can be achieved starting from the element stiffness matrix of the finite strip method. (More exactly: that version of the FSM which is included in the CUFSM software, which assumes sinusoidal displacement distribution along the member length.)

The following steps should be performed:

1) In a usual finite strip method the element stiffness matrix is an 8×8 matrix, since there are 4 DOFs per node: the 1 \((V)\) longitudinal displacement and the 3 \((U, W\) and \(θ)\) transverse displacements. What we are concentrating here is the transverse DOFs only, consequently, the rows and columns of the finite strip stiffness matrix associated with the longitudinal DOFs should be eliminated.

2) According to the applied GBT strain assumptions, there should be no membrane shear and membrane transverse \((x\text{-dir})\) strains. Consequently: all the terms that contain the shear modulus \(G\) should be eliminated. (Note: in-plane terms with longitudinal modulus \(E_2\) are already eliminated by the previous step.)

3) The original finite strip element stiffness matrix contains a number of terms that represent the coupling of longitudinal and transverse displacements \((\text{out-of-plane displacements})\). In fact, there are longitudinal displacements of the member. However, according to the applied GBT assumption, the transverse forces/stresses must be in equilibrium, which assumption can conveniently be interpreted as if the longitudinal displacements could take place freely, without developing stresses \((\text{or: as if there were no longitudinal displacements at all})\), which, in practice, can be expressed by the elimination of all these terms. It may be interesting to mention that all the terms to neglect contains the parameter \(m\) \((\text{or} \ km)\) which is the number of the assumed half sine-waves along the member length. Thus, we should eliminate all the terms with \(m\) \((\text{or} \ km)\).

4) By performing the previous steps we basically reach the same stiffness matrix as in Eq. (3.27) with the only difference that the elements of the above \(k_{\text{plate}}\) matrix are exactly the double of the corresponding elements of the finite strip matrix. The reason of the difference is the difference between the assumed \(Y\)-directional displacements. If a beam theory is applied for plates, as it is the case in the above considerations, there is the implicit assumption of \emph{constant} displacements in the beams transverse direction \((\text{which is the longitudinal direction of the real member, that is} \ Y\text{-directional})\). At the same time, the assumed \(Y\)-directional displacement distribution of the CUFSM software is \emph{sinusoidal}. Without detailed derivations it is obvious that a sinusoidal distribution is associated with smaller terms of the stiffness matrix, this latter having the physical meaning of stress resultants due to the unit displacement at the various DOFs. Moreover, considering the derivations of the stiffness terms, it is easy to show that the constant distribution means exactly twice as much stiffness as the sinusoidal one. Thus, if we want to have an element stiffness matrix exactly identical with the one derived from simple beam theory, we should multiply all terms of the finite strip element stiffness matrix by 2. (It is to mention, however, that for our purpose this last step has no practical importance since what is really important is not the real stiffness values but rather the stiffness distribution.)
3.3.3 Global stiffness matrix

To assemble the global stiffness matrix, there are basically the following steps to perform:

- co-ordinate transformation,
- compilation.

a) Co-ordinate transformation

According to the assumed local and global co-ordinate systems, the \( y \) and \( Y \) axes coincide. It means that the necessary co-ordinate transformation is a rotation about the member’s longitudinal axis, in the x-z or X-Z plane. This rotation can be used by the proper application of the rotation matrix, presented in Eq. (2.17). By considering the assumed order of the local DOFs, the transformation matrix can be written as:

\[
T = \begin{bmatrix}
\cos \alpha & \sin \alpha & 0 & 0 & 0 & 0 \\
-\sin \alpha & \cos \alpha & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & \cos \alpha & \sin \alpha & 0 \\
0 & 0 & 0 & -\sin \alpha & \cos \alpha & 0 \\
0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}
\]  

(3.28)

where \( \alpha \) is the angle of the given beam segment to the positive \( X \) axis. Now the element stiffness matrix in the global co-ordinate system can be calculated as:

\[
k_{g,\text{plate}} = T^T k_{l,\text{plate}} T
\]

(3.29)

b) Compilation

Compilation means the assemblage of the global stiffness matrix based on the element stiffness matrixes (transformed to the global co-ordinate system) and can be performed as usual in any finite strip or finite element program.

3.3.4 Nodal rotations and translations

a) Assumed order of DOFs in the global stiffness matrix

The order of the DOFs in the global stiffness matrix has no theoretical importance. Nevertheless, a properly selected order may make our expressions simpler. For this reason we introduce here a special DOF order which will be used in other places of this Report, too. The assumed DOF order is presented in Figure 3.3.
b) Transverse stiffness matrix

Since the problem in hand is the equivalent beam problem, it is enough to deal with those DOFs which are participating in the equivalent beam, too. These DOFs are the transverse DOFs, namely: \( X \) and \( Z \) displacements and the nodal rotations. Other words it means that the longitudinal displacements can be neglected in the equivalent beam problem which, furthermore, means that it is enough to consider a certain partition of the global stiffness matrix with those terms which correspond to the transverse DOFs.

The elimination of the longitudinal stiffness terms are illustrated in Figure 3.4.

\[
\begin{bmatrix}
\nu_1 \\
\vdots \\
\nu_{nm} \\
\dot{U}_l \\
\vdots \\
U_{nm} \\
\dot{W}_l \\
\vdots \\
W_{nm} \\
\theta_1 \\
\vdots \\
\theta_{nm} \\
\dot{U}_{ls} \\
\vdots \\
U_{ns} \\
\dot{W}_{ls} \\
\theta_1 \\
\vdots \\
\theta_{ns} \\
\nu_{ns}
\end{bmatrix}
\]

longitudinal displacements for all the main nodes

X-dir. displacements for the internal main nodes

Z-dir. displacements for the internal main nodes

X-dir. displacements for the external main nodes

Z-dir. displacements for the external main nodes

rotation for all the main nodes

X-dir. displacements for the sub-nodes

Z-dir. displacements for the sub-nodes

rotation for the sub-nodes

longitudinal displacements for the sub-nodes

**Figure 3.3: Assumed order of global Degrees of Freedom**
c) Expression for the unknown DOFs

The static equilibrium of the equivalent beam can be written as:

\[ \mathbf{K}_t \mathbf{d}_t = \mathbf{q}_t \]  \hspace{1cm} (3.30)

where \( \mathbf{K}_t \) is the transverse stiffness matrix, \( \mathbf{d}_t \) is the displacement vector for the transverse DOFs, while \( \mathbf{q}_t \) is the vector of nodal forces for the same DOFs.

According to the assumptions of the problem no external forces are applied, but the translations of the internal main nodes are prescribed. It means that the \( X \) and \( Z \)-directional displacements of the internal main nodes are supposed to be known, while all the other displacement DOFs are unknown. Let us partition Eq. (3.30) according to the known and unknown DOFs! (Note the DOF order is defined so that the known DOFs are just at the beginning of the transverse DOFs.)

\[
\begin{bmatrix}
K_{t,kk} & K_{t,ku} \\
K_{t,uk} & K_{t,uu}
\end{bmatrix}
\begin{bmatrix}
d_{t,k} \\
d_{t,u}
\end{bmatrix}
=
\begin{bmatrix}
q_{t,k} \\
q_{t,u}
\end{bmatrix}
\]  \hspace{1cm} (3.31)

where the “\( k \)” and “\( u \)” indexes correspond to the “known” and “unknown” DOFs.

Thus, \( \mathbf{d}_{t,k} \) is that part of the displacement vector which contains the known displacements of the internal main nodes, while \( \mathbf{d}_{t,u} \) contains the other DOFs which are unknown, and the determination of which is the main goal of this Sub-Section.

Similarly, \( \mathbf{q}_{t,k} \) is that part of the vector of nodal forces which contains forces acting on the known DOFs, while forces acting on the unknown DOFs is grouped in the \( \mathbf{q}_{t,u} \) partition of the force vector. Since external loads are not applied on the equivalent beam, this latter \( \mathbf{q}_{t,u} \) is definitely equal to zero. However, \( \mathbf{q}_{t,k} \) is not necessarily zero, since at the location of the prescribed (known) displacements kind of reaction forces may arise in order to fulfill the static equilibrium. Thus, Eq. (3.31) may be re-written as:

\[
\begin{bmatrix}
K_{t,kk} & K_{t,ku} \\
K_{t,uk} & K_{t,uu}
\end{bmatrix}
\begin{bmatrix}
d_{t,k} \\
d_{t,u}
\end{bmatrix}
=
\begin{bmatrix}
q_{t,k} \\
0
\end{bmatrix}
\]  \hspace{1cm} (3.32)
The first and second rows of the above equations can be regarded as two matrix-equations, as follows:

\[ K_{t,kk}d_{t,k} + K_{t,ku}d_{t,u} = q_{t,k} \]  \hspace{1cm} (3.33)

\[ K_{t,uk}d_{t,k} + K_{t,uu}d_{t,u} = 0 \]  \hspace{1cm} (3.34)

By the re-arranging of the second matrix-equation, that is Eq. (3.34), it is clear that it can directly be used to express the unknown displacements:

\[ K_{t,uu}d_{t,u} = -K_{t,uk}d_{t,k} \]  \hspace{1cm} (3.35)

Thus, \( d_{t,u} \) can formally be written as:

\[ d_{t,u} = -K_{t,uu}^{-1}K_{t,uk}d_{t,k} \]  \hspace{1cm} (3.36)

Some comments:

- Eq. (3.36) defines basically the same relationship which is defined by Eq. (2.75), Eq. (2.84), Eq. (2.91), Eq. (2.95) and Eq. (2.97), where the other unknown DOFs are expressed in function of the known \( U, W \) DOFs of the internal main nodes (or: nodes 2 to \( n-1 \)). However, Eq. (3.36) is not only simpler but even more general, since it can be applied for multi-branch cross-sections and cross-sections with sub-nodes, too. Nevertheless, if it is applied according to the restrictions of Section 2, it certainly leads to the same results as the corresponding equations of Section 2.

- Although Eq. (3.33) is not used, this equation has also its physical meaning. After solving Eq. (3.34), \( d_{t,u} \) can be calculated and substituted into Eq. (3.33) which, now, can be solved for the only unknown \( q_{t,k} \) vector. Thus, the transverse forces at the location of internal main nodes can be calculated, those which are necessary to maintain the transverse equilibrium.

**d) Matrix inversion**

In practice, it is not necessary to perform the matrix inversion of Eq. (3.36), although the \( K_{t,uu} \) matrix is certainly invertible. Instead, it is enough to solve a system of linear equations which is by far much simpler a problem mathematically than a matrix inversion.

To prove this, let us start with the fact that what we are really interested in is the \( -K_{t,uu}^{-1}K_{t,uk} \) matrix, since this is the matrix that defines the relationship between the known and unknown DOFs. It is easy to understand, however, that this expression can formally be regarded as the solution of a linear equation system as follows:

\[ K_{t,uu}X = -K_{t,uk} \]  \hspace{1cm} (3.37)

where \( K_{t,uu} \) is the coefficient matrix, \(-K_{t,uk}\) represents the multiple right-hand side, while \( X \) denotes the (multiple) unknown vectors. The formal solution for the above equation can be written as:

\[ X = -K_{t,uu}^{-1}K_{t,uk} \]  \hspace{1cm} (3.38)

thus, the unknown \( X \) matrix is just the matrix what defines the relationship between the known and unknown DOFs as explained above. However, to get this \( X \) matrix, it is not necessary to practically perform the matrix inversion, since more efficient methods are available.
4 Base vectors, pure modes, classification

4.1 Search spaces

a) General

Generally speaking the buckling analysis of the member, in fact, is the search for specific member displacements (and the associated critical forces) which satisfy certain conditions (equilibrium, compatibility, potential energy minimum, etc.). Since a member (either thin-walled or not) can deform infinite number of ways, the problem in hand is infinite-dimensional, that is: infinite number of different solutions may be found, theoretically.

Although it is rarely stated explicitly, all calculation method is looking for the solution of the member buckling problem in a smaller search space which is typically finite-dimensional. The characteristics of the search space are determined by the method and its assumptions (typically: assumptions regarding to the possible member displacements).

As a simple example the analytical solution for the pinned-pinned column buckling problem can be mentioned which is the most classical stability problem first solved by Euler more than 200 years ago, at least for flexural buckling. If we consider this flexural buckling problem, the solution, in practice, is searched for in a single-dimensional search space which is determined by some assumptions, as follows:

- As a first assumption, the assumption of rigid cross-section may be mentioned, which makes possible to describe the displacements of any cross-section by three scalars: two translations and the rotation. More general, the displacement field of the member can be described by three functions which define the above-mentioned three displacements along the member length.

- Next, the longitudinal distributions may be assumed as sine functions.

- Furthermore, if we are interested in the first buckling modes and critical forces only (which is typically the case in design), the longitudinal displacement distributions can be assumed to be half sine-waves, which reduces the search space into a special 3-dimensional space.

- Finally, if the displacements are assumed to occur in a specific plain, there is only one single half sine-wave, which is the only possible displacement mode of the member. In other words the search space is reduced to only one dimension.

Based on the previous Sections, however, a much more complicated example may also be mentioned, since Section 2 and 3 presents how the assumptions of the GBT makes possible to express certain DOFs in function of other DOFs, which in fact, means the reduction of effective DOFs, or, in other words, means a special selection of search space.

The above examples suggest that:

- the various calculation methods (analytical, GBT, FSM, FEM) work on various search spaces;

- the search space of a certain calculation method is defined by the basic assumptions of the method;

- it is the specially defined (reduced) search space that makes a method capable to directly give displacement patterns and critical loads/stresses associated with a certain buckling mode.
b) Basic hypothesis

As a generalization of the above idea, the following hypothesis is introduced here:

- The search space of the buckling problem of a thin-walled member can be sub-divided into four sub-spaces, which sub-spaces are defined by the special displacement-deformation pattern that is allowed in the given sub-space:
  - G sub-space which contains displacement-deformation patterns that correspond to the global buckling modes,
  - D sub-space which contains displacement-deformation patterns that correspond to the distortional buckling modes,
  - L sub-space which contains displacement-deformation patterns that correspond to the local buckling modes,
  - O sub-space that contains all the other displacement-deformation patterns which are not included in any of the above sub-spaces.

- It is also assumed that the G, D, L and O sub-spaces do not overlap, but cover all the search space (as indicated in Figure 4.1).

- The G, D and L sub-spaces can be defined by properly chosen assumptions for the deformations-displacements.

It is worth to mention that the above assumptions do not mean that the G, D, L and O sub-spaces are constants and can be defined independently of the method. On the contrary, they are typically dependent on the applied calculation method and also on the parameters of the method such as the applied finitization in case of FEM or FSM. However, in case of a given method and given parameters the division of the search space into G, D, L and O sub-spaces is assumed to be unambiguous.

Figure 4.1: Sub-division of the search space of a member buckling problem
c) **G and D sub-spaces**

Since the GBT is the only known method which is able to produce solutions for all the buckling modes (global, distortional and local), and, more importantly, it is able to produce them separately from each other, the assumptions of GBT seems to be applicable to define the sub-spaces.

As already mentioned in the earlier Sections, GBT has the following basic assumptions:

1) The membrane (in-plane) shear strains and membrane transverse (x-directional) strain is zero, as given by Eqs. (2.3) and (2.4).

2) The transverse moments, forces, stresses alone must be in equilibrium independently of the longitudinal ones, as it was already stated and used in Sections 2.4 and 3.3.

It must be noted that these above assumptions are valid only for the global and distortional modes, since local modes are handled separately even in GBT. At the same time, however, it is assumed that the above assumptions fully determine the G and D sub-spaces.

Moreover, to be able to distinguish between global and distortional modes (or, other words, to further sub-divide the GD sub-space into G and D), we adopt what GBT suggests.

To do this, first, we should recall what is shown in Sections 2 and 3, that the introduction of assumptions 1) and 2) reduces the search space in such a way that the member displacements can be expressed as a function of the longitudinal displacements. In many cases, the function of longitudinal displacements is referred as warping function, thus, we may say that the member displacements are fully determined by the warping function if the above two assumptions are applied.

Concerning the warping function, there is a further assumption, as follows:

3) The warping function is assumed to be a continuous function which must be linear within each plate element of the cross-section.

Considering assumption 3), it is obvious that the warping function can be described by its nodal value alone, which means that the Degrees of Freedom of the GD space is identical with the number of nodal lines \( n \) of the cross-section. (At least for single-branch cross-sections. For multi-branch cross-sections the number of DOFs is less than \( n \).) Thus, the GD sub-space can be described by maximum \( n \) independent (or: orthogonal) base functions (or vectors), containing the nodal displacements.

Now, the separation of G and D modes (or: sub-spaces) can be done as follows:

4) The G sub-space is that part of the GD sub-space which can be described by four special warping functions associated with special loads, namely: (i) pure axial load, (ii) pure bending moment about the first principal axis of the cross-section, (iii) pure bending moment about the second principal axis, (iv) pure torque (moment about the longitudinal axis). The D sub-space is the rest of the GD space. (The typical warping functions for global modes are illustrated in Figure 4.2 on a C cross-section.)

It is to mention that the assumption 4) is resulted in deformed shapes with no cross-sectional distortion, thus, from this point of view, it fully corresponds to the traditional definition of global buckling modes with rigid-body cross-section displacements. However, according to this GBT-like global mode definition, there are four global modes instead of the traditionally assumed three. It is an open question at this point whether the constant warping mode should be considered as global mode or not.
Another remark is that the warping functions associated with global modes are independent, or: orthogonal, which can easily be proven by mathematical or mechanical considerations. (The question of orthogonality will be discussed later in more detailed way.)

It should be noted that the above definition fails for cross-sections with less than three plate elements (less than four nodal lines), when the total number of DOFs is less than 4. This problem will be addressed later in this Report.

![Longitudinal displacement distributions for global buckling modes](image)

**Figure 4.2: Longitudinal displacement distributions for global buckling modes**

d) **L modes**

Local buckling modes are usually considered as modes involving plate flexure alone. In other words we may say that local modes should be those the deformation-displacement patterns of which correspond to that of the classical plate theory, since this latter does not consider any membrane (in-plane) deformation in any direction. From this point of view, the classical plate theory satisfies the GBT assumption 1) (for the nullity of certain membrane strains) as defined previously. However, it is easy to understand that it does not satisfy GBT assumption 2) (for transverse equilibrium), since:

- plate flexure type deformation always include longitudinal flexure and bending which is excluded from GD sub-space,
- GD modes always include longitudinal displacements, consequently longitudinal strains, which are assumed to be zero in plate buckling problems.

Thus, it seems equivalent to the above-mentioned local buckling mode definition to say that local modes are the ones that satisfy the GBT assumption for the nullity of some membrane strains but does not satisfy the transverse equilibrium.

e) **O modes**

Depending of the applied calculation method, other modes are possible. Thus, all the modes which cannot be categorized as global, distortional or local, should be categorized as other. Since all the G, D and L modes satisfy the GBT assumption for membrane strains, it seems convenient to define the O sub-space as the sub-space with all the deformation-displacement patterns where either the shear or the transverse membrane strains are not zero (or none of them, of course).
4.2 Base vectors for the buckling modes

4.2.1 Preliminary steps

Before starting the construction of base vectors, it is necessary to perform some preliminary calculations which will be used in defining the base vectors.

1) We should define the type of each node. The possible types are: (i) internal main node, (ii) external main node, (iii) sub-node.

2) The number of internal and external main node and the sub-nodes may be denoted as: \( n_{mi} \), \( n_{me} \), \( n_{s} \), respectively, while the total number of main nodes is \( n_{m} \).

3) Among the main nodes, we should select the determining main nodes (see Section 3.2) which are used to determine the (i) transverse translations of all the main nodes, and (ii) the longitudinal displacements of the un-determining main nodes. Note the determining main nodes can typically be selected in more than one way.

4) The number of determining and un-determining main nodes will be denoted by \( n_{md} \) and \( n_{mu} \), respectively.

4.2.2 Global modes

As it was shown before, the deformation-displacement pattern of global modes can be expressed in function of the longitudinal displacements (or: warping function), which is assumed to be linear along each plate element of the cross-section. Moreover, the warping functions must correspond to the linear combination of the ones presented in Figure 4.2. This definition provide with a convenient way for constructing the base vectors. (The procedure is illustrated in Figure 4.3.)
1) First, we should define the longitudinal nodal displacements according to the 4 (or 3, if the uniform warping is excluded from the G sub-space) pre-defined patterns. These values define the first \( n_{m} \) element of the global base vectors. Note this is the distribution what is important, thus, the real nodal values can arbitrary be scaled.

2) Next, the transverse translational displacements (namely: \( X \)- and \( Z \)-directional translations) of the internal main nodes can be defined. To do this we should apply Eqs. (2.33) and (2.34), if the cross-section fulfills the restrictions of Section 2. In a more general case, Eq. (3.12) must be used plate by plate. (Anyhow, this latter equation can always be used, even for simple cases.) This step defines the next \( 2 \times n_{mi} \) elements of the base vector.

3) The next step is the calculation of the other transverse DOFs, namely: (i) the \( n_{m} \) rotations of the main nodes, (ii) the \( 2 \times n_{me} \) \( X \)- and \( Z \)-dir translations of the external main nodes, (iii) the \( 3 \times n_{s} \) \( X \)- and \( Z \)-dir translations and rotations of all the sub-nodes. To do this there are at least two alternatives: 
   alt 1: Knowing that the global buckling modes allow no cross-sectional distortion, and considering that the \( X \)- and \( Z \)-dir translations of the internal main nodes are known from the previous step, all the other transverse displacements can be determined by linear interpolation (extrapolation).

---

**Figure 4.3: Procedure for defining global base vectors**
alt 2: Eq. (3.36) can conveniently applied, since this equation defines the relationship between the “known” (that is: known from the previous step) and “unknown” (that is: to be determined in this step) DOFs.

4) The only DOFs remained undetermined are the \( n_s \) longitudinal displacements of the sub-nodes. Again, there are at least two alternatives.
   alt 1: Since the longitudinal displacements of all the main nodes are known from the first step, and considering that the distribution is assumed to be linear between two neighboring main nodes, the \( V \) DOFs of the sub-nodes can easily be calculated by linear interpolation.
   alt 2: The \( V \) DOFs of the sub-nodes can be calculated in the same way as of main nodes. (Note that this step is independent from steps 2 and 3, thus, it can be done any time after the first step.)

5) The last step is the normalization of the base vectors defined by steps 1 to 4. As it will be shown later the normalization is necessary for buckling mode classification purposes only, since the sub-space is fully determined by any non-normalized base vectors. Another note that there are plenty of ways to normalize the base vectors, which question will be discussed separately.

### 4.2.3 Distortional modes

Similarly to global modes, in case of distortional modes the deformation-displacement field of the member can be expressed in function of the longitudinal displacements (or: warping function), which is assumed to be linear along each plate element of the cross-section. Moreover, the warping function must be orthogonal to those of the global modes, which can directly be utilized to define the distortional base vectors.

To keep the procedure general, we assume here multi-branch cross-sections which require somewhat special treatment due to the existence of defining and un-defining main nodes. In practice, an additional step is necessary to define the longitudinal displacements of the un-defining nodes from those of the defining ones. (The procedure is illustrated in Figure 4.4.)
1) First, we should define the longitudinal nodal displacements of the defining main nodes from the condition that the resulting warping functions must be (i) orthogonal to all the 4 global warping patterns, and (ii) orthogonal to each other. (Typically, this can be done in many ways, however, the sub-space, which is determined by these vectors, is always the same.) These values define the first $nmd$ element of the global base vectors. Note at this step the distribution is important, thus, warping functions can arbitrary be scaled.

2) Next, the longitudinal displacements of the other, un-defining main nodes should be calculated, by applying Eq. (3.16) plate by plate. (Note this step, in fact, cannot be performed separately from the previous one, since the application of orthogonality condition requires the knowledge of all the warping function, consequently, the knowledge of all the nodal values of the longitudinal displacements. Thus, the two first steps should be done simultaneously, and here separated only for the sake of easier understanding.) By the end of this step the first $nm$ elements of the base vector are defined.

Figure 4.4: Procedure for defining distortional base vectors

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69
3) Next, the transverse translational displacements (namely: X- and Z-directional translations) of the internal main nodes can be defined, exactly the same way as for global base vectors. We should apply Eqs. (2.33) and (2.34), if the cross-section fulfills the restrictions of Section 2, or in a more general case, Eq. (3.12) must be used plate by plate. This step defines the next $2 \times nmi$ elements of the base vector.

4) Similarly to global base vectors, the next step is the calculation of the other transverse DOFs, namely: (i) the $(nm)$ rotations of the main nodes, (ii) the $(2 \times nme)$ X- and Z-dir translations of the external main nodes, (iii) the $(3 \times ns)$ X- and Z-dir translations and rotations of all the sub-nodes, which can be done by applying Eq. (3.36).

5) Finally, the $ns$ longitudinal displacements of the sub-nodes should be defined. Since the longitudinal displacements of all the main nodes are known from the first and second steps, and considering that the distribution is assumed to be linear between two neighboring main nodes, the $V$ DOFs of the sub-nodes can be calculated by linear interpolation.

6) The last step is the normalization of the base vectors.

### 4.2.4 Local modes

To define base vectors for the sub-space of local buckling modes, we have to find all the independent vectors that:

- satisfy the GBT assumption for the nullity of membrane transverse and membrane shear strains,
- do not satisfy the equilibrium of transverse forces/stresses.

As it was previously discussed, this latter criterion also means that all the membrane strains are considered zero, while there are flexural (plate) deformations both in transverse and longitudinal directions. Furthermore, it means that the fold-lines of the cross-section remain straight.

Considering all these, in practice, the local buckling modes must satisfy the following criteria:

- All the longitudinal displacement DOF must be equal to zero $(nm+ns)$ DOFs,
- The transverse translation ($X$- and $Z$-dir) of the internal main nodes must be equal to zero $(2 \times nmi)$ DOFs,
- All the external main nodes and sub-nodes should not displace parallel to the adjacent plate elements (in other words: the local $u$ displacements are zero), which means $nme+ns$ DOFs.
- All the internal main nodes should be free to rotate. $(nmi)$ DOFs)
- All the external main nodes and sub-nodes should be free to rotate and free to displace perpendicularly to the actual plate element (other words: to displace in the local $w$ direction), which means $2 \times (nme+ns)$ DOFs.

The first three condition means that the corresponding elements of the local base vectors must be equal to zero, which means altogether $(nm+ns+2 \times nmi+nme+ns)$ DOFs, while the free DOFs can conveniently be handled by introducing a unit displacement from DOF by DOF, which means $nmi+2 \times (nme+ns)$ local base vectors. The physical appearance of the local base vectors are illustrated in Figure 4.5.
Some comments:

- The above defined local base vectors are obviously independent from the global and distortional base vectors. (E.g. consider the longitudinal DOFs.) It is also evident that they are independent of each other. Thus, the sub-space which they determine is unambiguously defined by them.

4.2.5 Other modes

The sub-space of other modes must contain all the possible deformation-displacement fields that are not included in none of the G, D and L sub-spaces. Thus, the dimension of this O sub-space as well as the way how the corresponding base vectors can be constructed is strongly dependent on the calculation method and its application. Hence, the other sub-space of the Finite Strip Method will be presented, although some of the ideas can directly be used for other methods, like FEM.

According to Table 4.1, the common feature of the G, D and L modes that the transverse membrane strains and the shear membrane strains are zero in all the plate elements that build the cross-section. At the same time, this condition does not stand for other deformation modes. Thus, independently of the calculation method, the sub-space of other modes should contain all the possible deformation-displacement fields where either the membrane shear or the membrane transverse strain is not zero (or both).

In case of FSM, both the membrane shear and the transverse membrane strain are assumed to be constant in the local x-direction within any plate elements (or, in other words: the functions of membrane strains are independent of x), since the $u, w$ displacements are assumed to be linear. It means that all the possible deformation-displacement fields can be defined by

**Figure 4.5: Local base vectors**
prescribing the value of the membrane shear and membrane transverse strain for each finite strip. This provides with a convenient way to define a system of independent base vectors, by simply applying unit strains for element by element. The resulting base vectors are illustrated in Figure 4.6 and Figure 4.7.

Figure 4.6. Other base vectors, with transverse membrane deformations

Figure 4.7: Other base vectors, with shear membrane deformations
4.2.6 Dimension of sub-spaces

Based on the definition of base vectors for the various sub-spaces, as described in the previous Sections, it is relatively easy to calculate the dimension number for the G, D, L and O sub-spaces, that is to define how many independent base vectors exist in each sub-space.

a) G sub-space

According to the GBT definitions, there are 4 independent global deformation mode, as illustrated in Figure 4.2, provided there are at least three plate elements (or: four nodal lines) in the cross-section. On the other hand, it is well-known that the classical analytical solutions for global buckling give maximum three global buckling modes. The difference between the analytical and GBT modes is the “axial mode” which is certainly is a possible global deformation, but certainly not a possible buckled shape. In order to be conform with the classical analytical solutions, it is proposed here not to include the axial mode among the global buckling modes. However, since the axial mode is certainly neither distortional nor local buckling mode, it will be counted as one of the other modes. It means that the number of the global buckling modes, that is the number of global base vectors is generally three.

It should be mentioned that the number of the real buckling modes is dependent also on the loading and boundary conditions, thus, the number of existing buckling modes may be less than the number of independent base vectors.

b) D sub-space

In case of single-branch open cross-section, it is evident that the number of external main nodes is always 2. \((nme = 2)\) It was also proved in Chapter 2 that the longitudinal displacements of all the main nodes are necessary to consider in order to express the other DOFs of the member, which means the number of determining main nodes is equal to the total number of main nodes. \((nmd = nm)\) Consequently, the number of independent G+D deformation modes (including the axial mode, too) is identical with the number of nodal lines. Since – in a general case – there are four global deformation mode, it is obvious that the number of independent distortional base vectors must be: \(nmd-4\).

For multi-branch cross-sections, as it was discussed in Chapter 3, the number of determining nodes is necessarily less the total number of main nodes. The difference depends on the number of nodes to which multiple plates are joined and the number of plates that are connected with a certain node. As it was proved in Section 3.2, for a multiple-plate node there are only two determining plates while the others are redundant, consequently, the nodes at the far end of such redundant plates are un-determining. Following this logic it is relatively easy to understand that the number of determining main nodes in a cross-section (can be either multi-branch or single-branch) is: \(nmd = nmi + 2\). (It is, of course, in full agreement with what previously said for single-branch cross-sections.) At the same time, it is still true that the number of independent distortional base vectors is \(nmd-4\), therefore, it can also be expressed as: \(nmi-2\).

c) L sub-space

As far as the L base vectors are concerned their number is easy to count (and, in fact, they are already counted previously), since they must have include the rotation of all the internal main
nodes plus the rotation and one translation of all the external main nodes and sub-nodes. Therefore, the number of local base vectors is: \( n_{\text{mi}} + 2 \times (n_{\text{me}} + n_{\text{s}}) \).

d) \text{O sub-spaces}

The number of O base vectors, again, is simple to calculate, since they must include:

- the axial mode, which means one base vector,
- the transverse membrane deformations, which means \((n_{\text{m}} + n_{\text{s}} - 1)\) base vectors as the number of strips is always equal to the number of nodes minus one,
- the membrane shear deformations, which means, again, \((n_{\text{m}} + n_{\text{s}} - 1)\) base vectors.

Thus, the total number of O base vectors is: \(2 \times n_{\text{m}} + 2 \times n_{\text{s}} - 1\).

e) \text{Total number of base vectors}

The total number of base vectors is the sum of the base vectors of the sub-spaces, that is:

\[
3 + n_{\text{mi}} - 2 + n_{\text{mi}} + 2 \times (n_{\text{me}} + n_{\text{s}}) + 2 \times n_{\text{m}} + 2 \times n_{\text{s}} - 1 = 2 \times n_{\text{mi}} + 2 \times n_{\text{me}} + 2 \times n_{\text{m}} + 4 \times n_{\text{s}} = 4 \times (n_{\text{m}} + n_{\text{s}}),
\]

which is exactly equal to the total number of DOFs of the finite strip model.

f) \text{Exceptions}

As already mentioned, there are some cross-sections where the above definitions may fail.

Since the construction of the local and other modes is quite evident it is reasonable to say that the L and O base vectors can always be constructed as defined above, consequently, their number can always be calculated as given above. Moreover, since the total number of DOFs is defined by solely the number of nodes, the sum of the numbers of G and D base vectors must be equal to: \(n_{\text{mi}} + 1\).

Now, the only cases when there is a problem with the above-defined G and D base vectors are the cases with too few internal main nodes, namely: when \(n_{\text{mi}}\) is less than 2, since in these cases it is impossible to have the 3 global base vectors. (3 global vectors would have been associated with -1 distortional base vectors, which is evidently impossible). Therefore, if we exclude the cross-sections with one single plate elements, the only problematic cross-sections are the ones with one single internal main node such as L, T, X and similar cross-sections.

In fact, the problem of such cross-sections is originated in the assumed warping functions for the global modes and in the fact that the global base vectors are created from the warping functions. As it was stated in Section 4.1 and illustrated in Figure 4.2, it is almost always possible to define four global modes. It is evident, that the axial and the two bending modes are always existing, even in case of L, T, X cross-sections, however, the torque mode does not exist. More exactly, pure torsional buckling evidently exists, however, such global base vector cannot be constructed. The reason is that cross-sections with one internal main node have their shear center exactly at the location of the only internal main node, consequently, the warping function for pure torque is constantly zero over the whole cross-section. At the same time, it is easy to prove that the pure torsional buckling mode for such cross-sections will take place among the local modes.
4.2.7 Constraint matrices

a) Illustrative example

In order to easier understand what follows, first, let us consider a simple problem (actually, the possible simplest problem), that is a single plate element (or: single strip), and let us concentrate on the membrane deformations only. According to the FSM the membrane deformation field of a single plate can be described by four nodal displacements: \( u_1, v_1, u_2, v_2 \) which means that the problem is 4-dimensional. As already shown, the introduction of GBT basic assumptions (membrane shear and membrane transverse strains are zero) will reduce the number of DOFs to two, since the transverse \( u_1, u_2 \) displacements can be expressed in the function of the longitudinal ones, as follows.

\[
\begin{bmatrix}
  u_1 \\
  v_1 \\
  u_2 \\
  v_2
\end{bmatrix}
= \frac{1}{k_m} \begin{bmatrix} 1/b & -1/b \end{bmatrix}
\begin{bmatrix}
  v_1 \\
  v_2
\end{bmatrix}
\]

(4.1)

Note the above equation is actually the repetition of Eq. (2.12) here. With a little re-arrangement we may also write:

\[
\begin{bmatrix}
  u_1 \\
  v_1 \\
  u_2 \\
  v_2
\end{bmatrix}
= \begin{bmatrix}
  1/k_m b & -1/k_m b \\
  1 & -1 \\
  1/k_m b & -1/k_m b \\
  0 & 1
\end{bmatrix}
\begin{bmatrix}
  v_1 \\
  v_2
\end{bmatrix}
\]

(4.2)

The left-hand side of Eq. (4.2) is the \( \mathbf{d} \) displacement vector of the original problem (a single plate element without the GBT assumptions), while the \( \mathbf{d}_r = \{v_1, v_2\} \) vector on the right-hand side is the displacement vector of the reduced DOF problem (single plate element which satisfies the GBT assumptions). Thus, Eq. (4.2) may be written in short as:

\[
\mathbf{d} = \mathbf{Rd}_r
\]

(4.3)

with the \( \mathbf{R} \) matrix defining the relationship between the original and the reduced DOFs.

An important feature of the \( \mathbf{R} \) matrix should be highlighted here. To discover the physical meaning of its columns, let us substitute the \( \mathbf{d}_r = \{1, 0\} \) vector into Eq. (4.2)! It leads to:

\[
\begin{bmatrix}
  u_1 \\
  v_1 \\
  u_2 \\
  v_2
\end{bmatrix}
= \begin{bmatrix}
  1/k_m b & -1/k_m b \\
  1 & -1 \\
  1/k_m b & -1/k_m b \\
  0 & 1
\end{bmatrix}
\begin{bmatrix}
  1 \\
  0
\end{bmatrix}
= \begin{bmatrix}
  1/k_m b \\
  1 \\
  1/k_m b \\
  0
\end{bmatrix}
\]

(4.4)
Thus, the first column of the $R$ matrix gives exactly the plate displacements for the unit translation of $v_1$ (which is the first DOF in the reduced DOF system) when the GBT assumptions are fulfilled. Similarly, the second column of the $R$ matrix is identical with the displacement vector that belongs to the $v_2 = 1$ displacement when the GBT assumptions are fulfilled. In other words the columns of the $R$ matrix can be regarded as base vectors of the deformations that satisfy the GBT assumptions. The two base vectors for this particular case are illustrated in Figure 4.8.

Having known that the columns of the constraint matrix are the base vectors of the reduced DOF problem, Eq. (4.3) can also be interpreted in a different way. The $Rd_r$ expression on the right-hand side can also be considered as a linear combination of the base vectors, where the elements of the $d_r$ vector are exactly the coefficients in the linear combination. However, this means that the resulting $d$ vectors are constructed as the linear combination of the base vectors, that is the $d$ vectors necessarily lay in the vector space defined by the base vectors. Since it is the $R$ matrix that forces the element to deform according to the GBT assumptions, it defines a certain constraint for the problem, thus, it is convenient to refer a matrix like $R$ as constraint matrix.

**Figure 4.8: Base vectors for the membrane deformations of a single plate element**

b) Application of the constraint matrices – example

Let us consider the static equilibrium of a single strip. The equilibrium equation can be written as:

$$Kd = f$$  \hspace{1cm} (4.5)

where $K$ is the stiffness matrix, $d$ is the displacement vector, and $f$ is the vector of nodal forces. By substituting Eq. (4.3) into Eq. (4.5), then pre-multiplying by $R^T$:

$$R^T KRd_r = R^T f$$ \hspace{1cm} (4.6)

The $R^T f$ expression on the right-hand side of the above equation is a vector with a length equal to the number of reduced DOFs. Physically, it is a force vector, the elements of which can be interpreted as the nodal forces corresponding to the reduced DOFs. It also means that the $R^T KR$ term on the left-hand side of Eq. (4.6) is the stiffness matrix associated with the reduced DOF problem. Thus, Eq. (4.6) may be re-written as:

$$K_r d_r = f_r$$ \hspace{1cm} (4.7)

with:

$$K_r d_r = f_r$$ \hspace{1cm} (4.8)
\[ f_r = R^T f \]
\[ K_r = R^T K R \]

(4.9)

c) Conclusions from the examples
The above theoretical examples demonstrated some application of the base vectors, also providing with some information on their physical meaning. As a summary, the following conclusions should be drawn:

- If we want to solve a problem in a special search space (that is: the DOF is reduced), what we need first is a system of independent base vectors of the special search space.
- By using the base vectors, it is possible to construct a constraint matrix, the columns of which should be the base vectors themselves.
- The relationship between the original and reduced DOFs is defined by Eq. (4.3).
- The relationship between the original and reduced vector of nodal forces is given by Eq. (4.8).
- The stiffness matrix of the reduced DOF problem can also be easily calculated, using Eq. (4.9).

4.3 Pure modes

4.3.1 Eigen-value problem in the reduced DOF space
With the application of the base vectors introduced above, it is possible to easily calculate critical loads (and the associated deformations) for any pure buckling modes or any combination of the pure modes.

Generally, when calculating critical loads (forces, etc.) the mathematical problem we face is a generalized eigen-value problem, can be written as follows:

\[ K d = \lambda K_g d \]

(4.10)

where \( K \) and \( K_g \) are the stiffness and geometric stiffness matrix of the member, \( d \) is an eigenvector (which, in fact, is a special displacement vector), while \( \lambda \) is the corresponding eigenvalue (the physical meaning of which is: load multiplier).

If we want to have pure buckling modes, basically the same type of eigen-value problem should be solved. The only difference is that we want eigen-vectors which satisfy not only Eq. (4.10), but can be expressed as the linear combination of the base vectors of the given mode. Thus, let us substitute Eq. (4.3) into Eq. (4.10)!

\[ K R d_r = \lambda K_g R d_r \]

(4.11)

then pre-multiply both sides by \( R^T \):

\[ R^T K R d_r = \lambda R^T K_g R d_r \]

(4.12)
It is easy to notice that in the above equation $R^T K R$ and $R^T K g R$ are the stiffness and geometric stiffness matrix of the reduced DOF problem, thus Eq. (4.12) may be re-written in a simpler form as:

$$K_r d_r = \lambda K_{gr} d_r$$  \hspace{1cm} (4.13)

where

$$K_r = R^T K R$$  \hspace{1cm} (4.14)

$$K_{gr} = R^T K g R$$  \hspace{1cm} (4.15)

Therefore, to have the buckling modes and critical loads for a specific buckling mode (e.g. global, distortional or local), the only modification that is necessary is the calculation of the reduced stiffness and geometric stiffness matrix by means of the constraint matrix, the procedure of the calculations otherwise is unchanged.

### 4.3.2 Constitutive equations for global modes

#### a) The problem

There is an interesting problem with the constitutive equations which arises when pure global buckling is calculated, which is discussed here.

In the analytical solution for global buckling the analyzed member is modeled as a beam. It means it is assumed that the length of the member is much larger than the cross-sectional dimensions, and, moreover, none of the width and height of the cross-section is considerably larger or smaller than the other. As a consequence, only the longitudinal normal stresses are considered, while the transverse normal stresses are assumed to be negligible. Thus, the stresses can be expressed as follows (by using a co-ordinate system with the y-axis parallel with the longitudinal axis of the member):

$$\sigma_x = 0$$

$$\sigma_y = E \varepsilon_y$$

$$\sigma_z = 0$$

Although it is rarely considered, it should be assumed that the transverse strains are not necessarily zero even if they are certainly small.

In the FSM calculations the constitutive equations for the plate problem can be expressed as follows, assuming isotropic material:

$$\begin{bmatrix}
\sigma_x \\
\sigma_y \\
\tau_{xy}
\end{bmatrix} = \begin{bmatrix}
\frac{E}{1-v^2} & \frac{vE}{1-v^2} & 0 \\
\frac{vE}{1-v^2} & \frac{E}{1-v^2} & 0 \\
0 & 0 & G
\end{bmatrix} \begin{bmatrix}
\varepsilon_x \\
\varepsilon_y \\
\gamma_{xy}
\end{bmatrix}$$  \hspace{1cm} (4.17)

where the notations are self-explaining. Note we will concentrate on the normal stresses only, the shear stresses/strains appear in the above equation for the sake of completeness.
As it is obvious from the above equation, it is implicitly assumed that the plates that construct the analyzed cross-section are thin enough (comparing to their width and length) so that the Poisson-effect should be considered. Thus, none of $\sigma_x$ and $\sigma_y$ is negligible, while $\sigma_z$ is implicitly assumed to be zero.

If we want to calculate the pure global modes by applying the constraint matrices as discussed above, we will automatically have deformations that satisfy the GBT basic assumptions, namely: $\varepsilon_x$ and $\gamma_{xy}$ is zero. Considering this, the normal stresses can be expressed as follows:

$$
\begin{align*}
\sigma_x &= \frac{vE}{1-v^2} \varepsilon_y \\
\sigma_y &= \frac{E}{1-v^2} \varepsilon_y \\
\sigma_z &= 0
\end{align*}
$$

The difference between the bar model (see Eq. (4.16)) and the constrained FSM model (see Eq. (4.18)) is conspicuous. For example, the Poisson coefficient for steel is approx. 0.3, which means that there is an approx. 10% difference between the longitudinal stresses of two model, which difference directly appears also in the bending rigidity of the member and finally in the critical load.

b) Discussion

Now the question is which model should be used when calculating pure global modes. In other words: which kind of constitutive equation gives more realistic results.

Although the bar model is a classical model of the structural mechanics, applied for centuries and at least for decades even in buckling problems, its application for thin-walled members can be regarded as an approximation only. It is enough to consider a simple symmetrical cold-formed C cross-section which, in a usual case, will buckle about its minor axis. Due to the minor axis bending, the majority of the elastic strain energy is coming from membrane strains/stresses, while the plates of the member are slender enough so that the Poisson-effect should not be neglected. Thus, if the bar model is used, the rigidity is under-estimated.

On the other hand, the constrained FSM model can also be criticized, especially if the results are compared to the probable real behavior. In reality it is unlikely that the cross-sections with such slender elements remain totally rigid when buckle. On the contrary, small plate-bending-type deformations take place, which can easily be demonstrated by a simple (unconstrained) FSM analysis. And these local plate deformations will reduce the global rigidity, that is why the constrained FSM model will over-estimate the unconstrained FSM results, and probably the real behavior, too.

Therefore, it seems there are two competing effects: the Poisson-effect and the local plate deformations.

- The simple bar model neglects both of them. More exactly, it neglects the local plate bending, and assumes that no transverse normal stresses arise during the deformations. This latter assumption also implies the presence of, even small, transverse strains.
- The unconstrained FSM model considers both the Poisson-effect and the local plate deformations. If the analyzed plate elements are thin enough (which is always the case for cold-formed steel), and the applied finitization is sufficiently dense, it is fair to say that
both effects are considered in a correct way, consequently, it is reasonable to assume that the unconstrained FSM model gives a good approximation of the real behavior.

- The constrained FSM model correctly considers the Poisson-effect but neglects the local plate deformations. It should be emphasized that not only the local plate bending is neglected, but also the in-plane deformations are assumed to be constantly zero, according to the basic GBT assumptions.

Not surprisingly, the results calculated with the different models will differ from each other. These differences clearly can be seen in Figure 4.9, where a simple C-section is analyzed.

---

\[ \frac{P_{cr}}{P_y} \]

---

**Figure 4.9: Global buckling with various models**

---

**c) Proposed solution**

As it is clear from the previous discussion, it is not evident which model could be regarded as a model which gives the correct values of critical load for global buckling. The problem in fact is originated in the not sufficiently precise definition of global buckling. Thus, the real question is how to define global buckling exactly. A few possible definitions are mentioned here as follows.

1) If the notion of “rigid cross-section” is strictly interpreted, neither the transverse flexure nor transverse extension of the plate elements is allowed.

In this case the constrained FSM model would be the correct model. However, both
theoretical considerations and numerical results suggest that the so-calculated critical loads are necessarily higher than the real ones. In addition, this interpretation is clearly in contradiction with the classical buckling solutions (e.g. Euler-formula).

2) Another option is to fully adopt the assumptions of the beam model. In this case the rigid cross-section would mean a cross-section without transverse bending, while the transverse extension/contraction is allowed. Moreover, it should be assumed that the transverse in-plane normal stresses are constantly zero in the whole member. Since this definition is in full agreement with the classical beam theory, the classical analytical buckling solutions should be considered as correct results. However, there is a big disadvantage of this definition: general numerical methods like FEM or FSM are practically not capable to fully reproduce these assumptions, since it seems impossible to ensure that transverse strains are free to arise with zero transverse in-plane stresses.

3) A good approximation of the analytical solution can be achieved with the constrained FSM model if the Poisson coefficient is set to zero. In this case the local plate bending is forced to be zero by the constraint matrix, which is in accordance with the analytical solutions. Moreover, the in-plane normal stresses are in full agreement with the analytical solution, since the transverse stresses are zero while the Poisson effect is eliminated from the longitudinal stresses, see Eq. (4.18). The only difference between this definition and the analytical one is in the transverse strain which is zero here while non-zero in the beam model. (It does not give any difference if there is one single plate element only, but is resulted in some differences if multiple plates are working together, which is always the case in practical cross-sections.)

According to the Author’s opinion the best definition is the last one, since it gives results that practically identical with the well-known analytical solutions, while it can be applied relatively easily within general methods like FEM or FSM. (Note the current implementation in the CUFSM software includes definition #3).

d) An important remark

There is an interesting fact which should be mentioned here. Although the applied constitutive equation has important effect on the critical loads for pure global buckling modes, as discussed above, it has no practical influence on the base vectors, consequently, on the constraint matrices.

As far as the G, L and O base vectors are concerned, they can evidently be constructed by using the cross-section geometrical data, thus, they are certainly independent of the material constants. On the other hand, the definition of the D base vectors involves material data, too, since Eq. (3.36) is used which includes a certain partition of the global stiffness matrix, consequently, includes material constants. However, more detailed analysis of the elements of the stiffness matrix in the referred equation shows that only the transverse bending stiffness elements have important role. More exactly, it is not the stiffness values that are important, but rather the transverse bending stiffness distribution. And this latter one is clearly independent of the value of the Poisson coefficient in every reasonable case. (The only exception would be the case of a cross-section where materials with considerably different Poisson coefficients are joined together, which is unlikely to happen in real situations.)
4.4 Modal decomposition

4.4.1 Orthonormal base vectors

a) General

Performing a modal decomposition of a general deformation mode means the calculation how the buckling modes contribute in the deformation. Mathematically the problem can conveniently be handled by (i) defining an orthonormal base system, then (ii) expressing the deformation-displacement pattern as the linear combination of the orthonormal bases. Although the procedure is simple in principle, its practical implementation requires the clarification of some questions which are discussed as follows.

b) Orthogonality conditions, normalization

Understanding the construction of base vectors as presented in the previous Sections, it is obvious that the G, D, L and O vector (or function) spaces are unambiguously defined by the assumptions which are summarized in Table 4.1. Moreover, it is also obvious that the defined base vectors are independent of each other, which means that none of them can be expressed as the linear combination of the others. Thus, the first question which should be clarified is the definition of orthogonality.

Theoretically, there are plenty of ways to define orthogonality, some of which are listed as follows.

1) A natural way is the orthogonality in vector sense. In this case two vectors are orthogonal if their scalar product is zero. In our case: two base vectors (which, in fact, are displacement vectors) would be orthogonal if their scalar product were zero. Moreover, the vectors could be normalized so that the scalar product would be equal to one.

\[
d_i^T d_j = \begin{cases} 
1 & \text{if } i = j \\
0 & \text{if } i \neq j 
\end{cases}
\] (4.19)

Although this interpretation is clear and simple, it would mean that the orthogonality, and especially the normalization significantly depends also on the applied finitization, which is a definite disadvantage.

2) Another option is to use the displacement functions (which are represented by the vectors, of course) instead of the displacement vectors to avoid the dependency of the orthogonalization on the applied finitization. The formula, somewhat symbolically, can be written as follows:

\[
\int [f(d_i) f(d_j)] = \begin{cases} 
1 & \text{if } i = j \\
0 & \text{if } i \neq j 
\end{cases}
\] (4.20)

Although this method is independent on the finitization, it has still some disadvantages. One major problem is that various functions are summarized in the integration, including functions of both translations and rotations. It may cause severe problem especially in the normalization, since scalars with different orders should be summarized which can easily lead to false results.

3) To avoid the above problem, it is possible to apply the integration for not all the displacement functions but only some of them. This idea, in fact, is used in the definition
of global and distortional base vectors, where it was stated that the function of longitudinal translation (that is: the warping function) of the global and distortional base vectors must be orthogonal. In that definition a formula similar to Eq. (4.20) was used, applied only for the warping function and being integrated above the cross-sectional area. It may not be evident, but can easily be proved, that the so-created G and D base vectors are not necessarily orthogonal in other sense.

4) It is well-known that the eigen-vectors themselves are orthogonal in a certain way. Thus, a theoretically simple way of defining an orthonormal base system is to use the eigen-vectors as base vectors. Since this is the solution that is proposed by the Author, it is discussed below in more detail.

c) Proposed orthonormal base vectors

Since it is crucial to keep the G,D,L,O sub-division of the base vectors, when the eigen-vectors are used as orthogonal base vectors, Eq. (4.13) must be applied separately in the G, D, L and O sub-spaces. This will be resulted in four orthogonal vector-systems. More exactly: the orthogonality is satisfied within each sub-space, while the vectors of a certain sub-space will not necessarily orthogonal to all the vectors of another sub-space.

Let us see what the orthogonality means for the eigen-vectors (which is necessary for the normalization)! Let us consider two different eigen-vectors (in the same sub-space, of course) with the corresponding eigen-values. These will be denoted by subscripts \( i \) and \( j \). Both of them satisfy Eq. (4.13), thus, we may write:

\[
K_r d_{r,i} = \lambda_i K_{gr} d_{r,i} \tag{4.21}
\]

\[
K_r d_{r,j} = \lambda_j K_{gr} d_{r,j} \tag{4.22}
\]

Let us pre-multiply Eq. (4.21) by \( d_{r,j}^T \)!

\[
d_{r,j}^T K_r d_{r,i} = \lambda_i d_{r,j}^T K_{gr} d_{r,i} \tag{4.23}
\]

Now let us transform Eq. (4.22), considering that both the stiffness and the geometric stiffness matrix are symmetrical:

\[
d_{r,j}^T K_r = \lambda_j d_{r,j}^T K_{gr} \tag{4.24}
\]

Let us multiply (from the right) the above equation by \( d_{r,i}^T \)!

\[
d_{r,j}^T K_r d_{r,i} = \lambda_j d_{r,j}^T K_{gr} d_{r,i} \tag{4.25}
\]

Subtracting Eq. (4.25) from Eq. (4.23) we may write:

\[
0 = (\lambda_i - \lambda_j) d_{r,j}^T K_{gr} d_{r,i} \tag{4.26}
\]

If \( i \) and \( j \) are different, the corresponding eigen-values are generally not equal to each other. It means that the above equation is satisfied only if:

\[
d_{r,j}^T K_{gr} d_{r,i} = 0 \tag{4.27}
\]

is satisfied. (On the other hand, if \( i \) and \( j \) are identical, \( d_{r,j}^T K_{gr} d_{r,i} \neq 0 \) is possible.)

Back-substituting Eq. (4.27) into Eq. (4.23) or into Eq. (4.25), it is obvious that
\[ d_{r,j}^T K_r d_{r,i} = 0 \]  \hspace{1cm} (4.28)

is also satisfied.

Thus, the orthogonality condition, which is automatically satisfied by the eigen-vectors of the generalized eigen-value problem, can be summarized as follows:

\[
\begin{align*}
    d_{r,j}^T K_r d_{r,i} & = \begin{cases} 
        \neq 0 & \text{if } i = j \\
        = 0 & \text{if } i \neq j
    \end{cases} \\
    d_{r,j}^T K_{gr} d_{r,i} & = \begin{cases} 
        \neq 0 & \text{if } i = j \\
        = 0 & \text{if } i \neq j
    \end{cases}
\end{align*}
\]  \hspace{1cm} (4.29)

\[ (4.30) \]

Eqs. (4.29) and (4.30) provide with a convenient way for the normalization, too. In addition, a meaningful physical interpretation can also be assigned to the suggested normalization by noticing that the expression on the left-hand side of Eq. (4.29) is exactly the double of the elastic strain energy accumulated in the member if a deformation defined by the displacement vector takes place. Since the accumulated strain energy has a easy-to-understand physical meaning, it is not sensitive to the finitization (especially if the finitization is fine enough), plus it can be properly (and easily) calculated independently of the method (FEM, FSM or other), assumed DOFs, etc., it can be regarded as a very convenient measure for the normalization of the eigen-vectors (that is: the orthogonal base vectors). Thus, the base-vectors will be scaled as follows:

\[ \frac{1}{2} d_{r,i}^T K_r d_{r,i} = 1 \]  \hspace{1cm} (4.31)

for all the base vectors.

By pre-multiplying Eq. (4.21) by \( d_{r,i}^T \) then dividing by 2, we may write:

\[ \frac{1}{2} d_{r,i}^T K_r d_{r,i} = \lambda_i \frac{1}{2} d_{r,i}^T K_{gr} d_{r,i} \]  \hspace{1cm} (4.32)

Back-substituting Eq. (4.31) into Eq. (4.32) we get:

\[ \frac{1}{2} d_{r,i}^T K_{gr} d_{r,i} = \lambda_i \]  \hspace{1cm} (4.33)

Finally, by considering Eqs. (4.29), (4.30), (4.31) and (4.33), we may summarize the proposed orthonormality condition as follows:

\[ \frac{1}{2} d_{r,j}^T K_r d_{r,i} = \begin{cases} 
    1 & \text{if } i = j \\
    0 & \text{if } i \neq j
\end{cases} \]  \hspace{1cm} (4.34)

\[ \frac{1}{2} d_{r,j}^T K_{gr} d_{r,i} = \begin{cases} 
    \frac{1}{\lambda_i} = \frac{1}{\lambda_j} & \text{if } i = j \\
    0 & \text{if } i \neq j
\end{cases} \]  \hspace{1cm} (4.35)
4.4.2 Calculation of mode contribution

Once the orthonormal base vectors are calculated, any displacement vector (that may belong to an arbitrary, possibly interacted buckling mode) can be expressed as the linear combination of the base vectors. Let us mark, hence, the orthonormal base vectors by an \( o \) subscript. Now, the problem can be written as follows:

\[
D_o c = d
\]  \hspace{1cm} (4.36)

where \( D_o \) is a square matrix constructed from the orthonormal \( d_o \) base vectors so that each column of \( D_o \) would be a base vector; \( d \) is the given general displacement vector, while \( c \) is a vector containing the coefficients of the base vectors in the linear combination. Of course, these are the elements of the \( c \) vector which are unknown and which are to be calculated. As it is obvious, the above equation is an inhomogeneous system of linear equations which, due to the orthogonality of the columns of \( D_o \) matrix, can always be solved.

After the \( c \) vector is calculated, the contribution of any individual mode can conveniently be determined as the ratio of the coefficient of that mode and the sum of all the coefficients, as follows:

\[
\frac{|c_i|}{\sum_{\text{all}}|c_i|}
\]  \hspace{1cm} (4.37)

Similarly, the contribution of a mode class can be defined as:

\[
\frac{\sum_{\text{mode}}|c_i|}{\sum_{\text{all}}|c_i|}
\]  \hspace{1cm} (4.38)

Note other formulae for the mode contribution calculation are also possible, for example based on Euclidean norm of the \( c \) vector and of its partitions, however, according to the Author’s opinion the above expressions fit the most to the application of linear combination, therefore, Eqs. (4.37) and (4.38) are implemented also in the CUFSM software.
5 Numerical examples

5.1 Examples for pure mode calculation

Hence, some numerical examples are presented in order to demonstrate how the method, which is presented in the previous Chapters of this Report, works in practice. In this Section, first, the pure more calculation is demonstrated. The examples correspond to the examples presented in Section 1.3, namely: Examples 4 to 9. Note here only the critical force charts are presented, the corresponding deformed cross-section shapes can be found in Section 1.3.

Figure 5.1 shows the critical forces in function of the buckling length for Example 4, which is a simple C-section. (Cross-sectional data are given in Section 1.3.1.) As it can be seen the first local, distortional and global buckling loads are presented together with the original FSM (all-mode) calculation. It should be emphasized here that all the curves were calculated by the CUFSM software, that is by using the Finite Strip Method, the pure mode curves, however, by applying the corresponding constraint matrices.

It is conspicuous that for smaller and larger buckling lengths the all-mode curve is identical with the pure local and pure global curves, respectively. For intermediate buckling lengths there is also a good agreement between the all-mode and the pure distortional curves, however, a small difference definitely exists. This difference suggests that, even if the all-mode curve has a distinct distortional minimum point, the real deformation mode is not a pure distortional mode but probably an interaction between the distortional and some of the other modes. (This interaction will be more clearly demonstrated later by the modal decomposition.) Note an even stronger interaction can also be observed around those buckling lengths where the pure mode curves intersect. The interaction, which is otherwise well-known from the literature, is marked by the fact that the all-mode curve lay considerably below the pure mode curves in these buckling length regions.

Figure 5.1: Pure buckling modes critical forces for Example 4
Figure 5.2 shows the same curves as in the previous example, however, for Example 5. The analyzed cross-section again is a simple C-section, the dimensions of which are given in Section 1.3.2.

Basically similar observations can be done as in the previous example, although the interaction is more remarkable now for intermediate buckling lengths. This strong interaction can certainly traced back to the lack of distortional minimum point on the all-mode curve, which is one of the major problem of the original FSM calculation. The important thing is, however, that the constrained FSM is capable to calculate the location and the value of the minimal critical force, the all-mode curve either has or has not a distortional minimum point.
Figure 5.3 to Figure 5.6 presents the critical forces for C-sections with small web stiffeners (Examples 6 to 9). In the case of the first three cases (Examples 6, 7 and 8) the figures do not substantially differ from each other, as the analyzed cross-sections hardly differ, neither. The characteristic feature is that the value of pure local critical force is high enough so that the local minimum disappears from the all-mode curves. This is why the first minimum points of the all-mode curves, in fact, belong to the distortional mode (slightly interacted with some other modes).

Figure 5.4: Pure buckling modes critical forces for Example 7

Figure 5.5: Pure buckling modes critical forces for Example 8
Unlike in Examples 6 to 8, in case of Example 9 the local minimum exists, too, on the all-mode curve. There are two more minimum points, both of them being a distortional minimum, as it is clear from the pure mode calculations.

![Figure 5.6: Pure buckling modes critical forces for Example 9](image)

**5.2 Examples for modal decomposition**

Two examples are presented in this Section in order to illustrate some results of modal decomposition. (Note other examples are found in the subsequent Sections.) The two examples are: Examples 5 and 9, the first being a simple C-section while the second is a C-section with a web stiffener.

Figure 5.7 and Figure 5.8 present the critical forces and the modal contributions as the function of buckling length. Although the charts for the two examples are definitely different, they have some common features, too. What is conspicuous is the similar tendencies.

- Local modes have practically 100% contribution for smaller buckling lengths, then their contribution decreases to practically zero, as the buckling length increases.
- Distortional modes have important contribution only at intermediate buckling lengths, which is clearly marked by the bell shape of the corresponding curve.
- The global modes contribution curve starts at zero than increases together with the buckling length up to 100%.
- The contribution of the other modes is negligible in both presented cases.

It should be mentioned, however, that, although the above tendencies seem to be typical in many practical cases, other tendencies may happen. Some examples will be shown below.

Another remark is that, as already mentioned in the previous Section, there is no pure distortional mode in case of Example 5. This fact is reflected both in the lack of distortional minimum on the critical force curve and, more clearly, in the distortional mode contribution curve which does not go higher than 60% in this particular case.
Figure 5.7: Modal decomposition for Example 5

Figure 5.8: Modal decomposition for Example 9
5.3 I-sections

As it has been mentioned in Chapter 1, it is a question whether the distortional buckling of I-sections exists or not.

Figure 5.9 presents the buckled cross-sectional shapes of an I beam in bending for various buckling lengths. By looking at the deformed shapes only, one might easily identify all the three modes. In this particular example the buckled shapes suggest that pure local buckling takes place until a buckling length equal to 200 mm (which is identical with the section depth), there is distortional buckling at approx. 800-1500 mm of buckling length, while there is pure global buckling (namely: lateral-torsional buckling) for lengths larger than 5-6000 mm.

At the same time, there are reasons to say that distortional buckling is not possible for I-sections.

- The all-mode buckling curve clearly has a minimum point at approx. 100 mm, which is definitely a local buckling minimum. However, any sign of a second, distortional minimum cannot be seen on the curve of the critical loads, see Figure 5.10.

- It was shown that, if we accept the mode definitions summarized in Table 4.1, the number of distortional modes is equal to the number of internal main nodes minus 2. Since an I-section has only 2 internal main nodes, the number of distortional modes is zero. This is, of course, reflected in Figure 5.10, where the pure curves are also plotted. And the lack of distortional modes can more evidently be seen in Figure 5.11 which shows the modal decomposition curves. This latter figure, on the other hand, indicates that the buckled shapes which are seemingly distortional (around 1000 mm) are, in fact, the results of a strong interaction of local and global modes.

![Figure 5.9: Buckled cross-sectional shapes for Example 3](image)
Figure 5.10: Pure mode critical curves for Example 3

Figure 5.11: Modal decomposition for Example 3
5.4 The effect of small stiffeners on the modal decomposition

In Chapter 1 it has been demonstrated that cross-sections with stiffeners may make the buckling classification problematic, especially if the stiffener is small. The problem now can even more evidently shown, by using the ability of modal decomposition.

As illustrative examples, Example 1 (a) and (b) are presented here, first. The two cross-sections are practically identical: cross-section (a) being a simple C-section (for the dimensions see Section 1.2.2) while cross-section (b) is a C-section with a very small web stiffener. In fact, the dimension of the web stiffener is almost in the order of geometrical imperfections, since its maximal dimension perpendicular to the web is only 1 mm, while the web depth is 200 mm.

As one might expect, the critical forces and deformed shapes of the two cross-sections are very similar (see Figure 1.3 and Figure 1.4), the difference in the critical forces is hardly can be seen in Figure 5.12: Pure mode critical curves for Example 1 (a) and (b). Consequently, it would be logical to expect that the classification of the buckling modes at various buckling lengths should be the same.

On the other hand, if we apply the definitions suggested by GBT (and used throughout this research report), the corresponding modal decomposition curves significantly differ from each other as it can be observed in Figure 5.13 and Figure 5.14. For the simple C-section (Example 1 (a)) the modal decomposition curves follow the usual tendencies: primarily local buckling for smaller buckling lengths, distortional for intermediate lengths and global for longer lengths (although the distortional contribution does not go higher than 80% even for intermediate lengths). However, as soon as there is an even small web stiffener, the buckling mode which involves significant web deformation becomes distortional mode, while the local mode almost disappears. For example, at the length of 170 mm (which is approx. the lengths of the local minimum on the curve of critical forces) there is a more than 99% contribution of the local modes for the C-section, while for the cross-section with the small web stiffener there is 5% and 94% contribution of the local and distortional modes, respectively.

![Figure 5.12: Pure mode critical curves for Example 1 (a) and (b)](image)
Figure 5.13: Modal decomposition for Example 1 (a)

Figure 5.14: Modal decomposition for Example 1 (b)
Exactly the same problem can be observed in Example 2, cross-sections (a) and (b), where cross-section (a) is a simple unlipped channel section (or U-section) while cross-section (b) is a very similar channel section, however, with a small end-lip. The curves of critical forces as well as the modal decomposition for both cross-sections are presented in Figure 5.15. (For buckled shapes, see Figure 1.7 and Figure 1.8.)

According to mode definitions, an unlipped channel section does not have distortional mode, since the number of internal main nodes is only 2. Accordingly, for smaller and intermediate buckling lengths a practically pure local buckling occurs until the global buckling becomes determinant. On the other hand, the presence of an even small end-lip increases the number of internal main nodes, thus, distortional modes appear. Moreover, due to the small rigidity of the small end stiffener, distortional modes are dominant for a wide range of buckling lengths, while pure local buckling may take place in case of very small buckling length only.

![Graph A](image1)

![Graph B](image2)

**Figure 5.15:** Critical forces and modal decomposition for Example 2 (a) and (b)
6 Summary

a) What is solved
This report summarizes the research activity on the buckling modes of thin-walled prismatic members with open cross-sections.

- We have proposed a method which enables general purpose numerical methods (such as FSM and FEM) to calculate pure buckling modes directly.
- The proposed method has been extended to be able to perform modal decomposition of a general deformation mode, that is to calculate the modal contribution from the local, distortional and global modes in a general buckling mode.
- In order to apply the proposed method, it was necessary to have clear definitions for the buckling modes. For this, the definitions suggested by the Generalized Beam Theory were formulated and adopted.
- The necessary derivations have been done in the context of the Finite Strip Method. More exactly the FSM as used in the CUFSM software has been used, which assumed sinusoidal displacement, deformation, etc. distribution in the longitudinal direction of the member. (Being so, it is appropriate to model pinned-pinned members, or such portions of members which correspond to pinned-pinned boundary conditions.)
- The method has been implemented in the CUFSM software.
- Numerical examples have been worked out to illustrate the capabilities of the proposed method and of the applied definitions for the various buckling modes.

b) Remaining problems
There are problems which can be regarded as unsolved problems.

- As it is clear from the numerical examples, the applied definitions for the local and distortional modes sometimes leads to mode classification which is in contradiction with the engineering expectations, as well as in contradiction with the current practice. As characteristic examples, the cross-sections with small stiffeners may be mentioned.
- For some cross-sections, such as an I-section, the existence of distortional modes is a question, too. By applying the GBT-suggested mode definitions, it is clear that distortional modes are not possible. On the other hand, one might identify distortional modes on the basis of another, phenomenological mode definition.
- As it was proved, the seemingly simple question of global mode definition involves some problems, too. One problem is the interpretation of “rigid cross-section”, more exactly: whether the transverse (in-plane) deformations are allowed or not.
- For cross-sections with zero warping function for pure torsion, the pure rotation of the cross-section may be classified both as global, both as local buckling mode. However, for modal decomposition, it cannot be allowed to have two identical modes, since it would mean two identical base vectors, which is a self-contradiction.
c) Future works

Shorter or longer-term plans may be mentioned to discover the abilities and limitations of the proposed method, as well as to extend it for more general problems.

- More numerical examples should be worked out to check the results, especially those of the modal decomposition, against the current practice and definitions.
- Numerical studies should be performed to analyze the effect of various orthogonalization and normalization of the base vectors.
- Numerical studies should be performed to analyze the importance of other modes. In the presented examples the modal contribution from the other modes is always negligible, but other examples suggest that other modes may have importance.
- Although interacted buckling modes are partially covered by the current design recommendations, the capability of calculating modal contributions in a general (interacted or coupled) buckling mode gives the possibility of developing more advanced design methods that more accurately handle the coupled modes.
- Since the Finite Strip Method has certainly some limitations, it would be extremely useful to extend the proposed idea for more general numerical methods, including FSM with other longitudinal shape functions and, more importantly, including the Finite Element Method. The successful application in FEM would give the ability of analyzing almost any members of practical importance (e.g. members with holes, etc.).
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