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Yield-line Analysis of Cold-formed Steel Members

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Abstract

The objective of this paper is to provide a state of the art review on the application of yield-line analysis to cold-formed steel members, and present a newly developed solution to the stresses that develop in an inclined yield-line. Yield-line analysis in cold-formed steel members is shown to be distinct from traditional yield-line applications. Challenges in the application of yield-line analysis include (1) the need for an *a priori* definition of the spatial collapse mechanics of interest and (2) the widely varying solutions that are provided in the literature for the bending strength of yield-lines. Based on the von Mises yield criterion, in the fully plastic state ($\nu = 0.5$), and assuming only normal and transverse stresses exist for bending about an inclined yield-line axis, new expressions are provided and verified for the stresses in an inclined yield-line. Although the assumed local stress state is verified, it is found that development of the correct load-deformation response requires a more complete treatment of the kinematics of the actual deformation and the developed forces. Finally, despite current challenges, future applications of yield-line analysis are discussed for the study of imperfection sensitivity, ultimate strength prediction, ductility, and energy absorption. All of which are important problems for future research in the behavior and design of cold-formed steel members.

Keywords: cold-formed steel, generalized yield-line analysis, spatial mechanism, collapse mechanism

1. Introduction

Two interesting nonlinear phenomena govern the behavior of cold-formed steel members under increasing load: (1) instability of the slender elements that comprise the section, and (2) localization of the inelastic response and formation of a spatial mechanism in the post-peak range as the member collapses. To understand the inter-play between elastic buckling and inelastic mechanisms consider the simple bar-spring model of Fig. 1. Response of the model includes both post-buckling stable and post-buckling unstable modes as well as multiple yielding mechanisms. Response of a cold-formed steel member is analogous, but includes more buckling modes and collapse mechanisms.

If the critical buckling mode has a stable, or neutral, post-buckling path, e.g., $P_{cr\phi}$ of Fig. 1c, then the member has potential to exhibit significant ductility and the collapse behavior should be governed by the mechanism response. Local buckling of cold-formed steel members typically fall in this regime. If the critical buckling mode has an unstable post-buckling path, e.g., P_{crx} of Fig. 1b, then ductility and collapse behavior is governed either by the mechanism response or by the elastic post-buckling response. Such members are likely to be strongly imperfection sensitive because of both the elastic post-buckling response and the response of the failure mechanism itself. Shell structures, and certain unfortunate frame assemblages, are known to exhibit this behavior.

Cold-formed steel members are typically composed of highly slender elements. As such, the behavior and design of these members from service to ultimate load is largely governed by the stability of the cross-section. Traditional design methods for cold-formed steel members, such as the widely used effective width method (Winter, 1947) or the more recently developed Direct Strength Method of the senior author (Schafer, 2002; NAS, 2004), recognize this by using relatively detailed models for elastic stability, but simplified models of the response due to yielding.

Particularly for design under extreme loads, when load re-distribution of cold-formed steel members needs to be considered, efficient methods are needed that examine the complete collapse response. Beyond testing, the standard recourse in such a situation is finite element (FE) solutions with material and geometric nonlinearity and employing incremental equilibrium path-following techniques (e.g., Riks). Unfortunately, FE methods following this approach are computationally costly, and by their very nature can not provide the type of analytical solutions which are useful in understanding the basic mechanics of collapse behavior. Yield-line analysis is an alternative method which attempts to provide only the relevant inelastic response, or mechanism curves of Fig.

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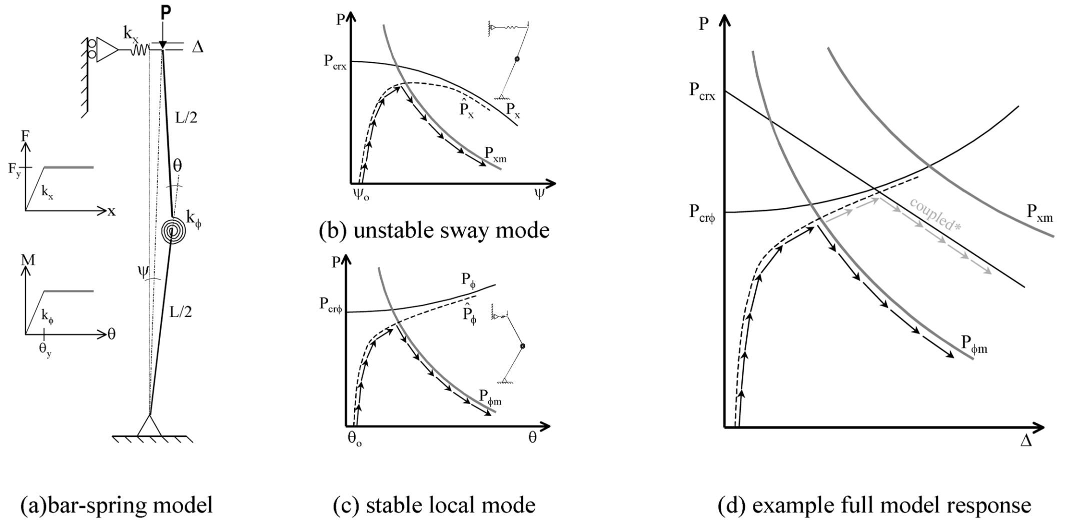


Figure 1. Mechanical model for demonstrating relevant thin-walled structural behavior. Fig. 1(d) as shown with the shaded arrows, if the mechanism (e.g. $P_{\phi m}$) does not occur until larger Δ, then the coupled instability: local-sway, which has unstable post-buckling behavior controls.

1. As detailed in the discussion section of this paper, numerous productive uses exist for a method which can generate a sound mechanism curve.

2. Spatial Mechanisms

In a cold-formed steel member, as localization of the inelastic response occurs, a spatial failure mechanism develops. In yield-line analysis these deformations are assumed to occur in zero-width lines: yield-lines. If cross-section distortion is not allowed (Fig. 2a), the derived mechanism is consistent with common plastic hinge models used in hot-rolled steel member design. If cross-section distortion is allowed (Fig. 2b), a spatial mechanism forms. Spatial mechanisms are required for approximating the collapse of thin-walled cold-formed steel members.

Spatial mechanism analysis can be broken into two groups (categorization after Bakker, 1990): classical and generalized (Fig. 3). Classical yield-line analysis, such as that to determine the ultimate strength of a transversally loaded concrete slab, assumes only primary (first-order) bending moments contribute to the energy dissipation in the mechanism. Generalized yield-line analysis provides the load-deformation relationship of the mechanism, and is driven by the second-order displacements that occur as the initial buckling deformation ensues.

Classical yield-line analysis is important because (1) it has been fully developed and (2) it is synonymous with the phrase “yield-line analysis” in the bulk of the literature. Ultimate strength prediction of transversally loaded concrete slabs is the most common application. The analysis method traces its origins to Ingerslev (1923) and is now accepted in concrete design specifications worldwide. The method has also been incorporated into FE models (Munro and DaFonseca, 1978) and significant

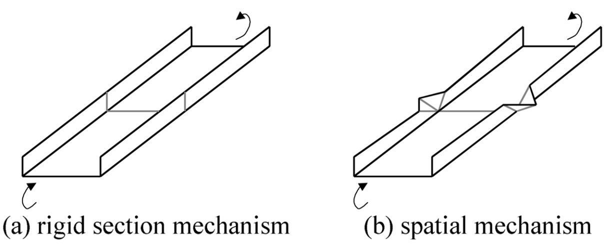
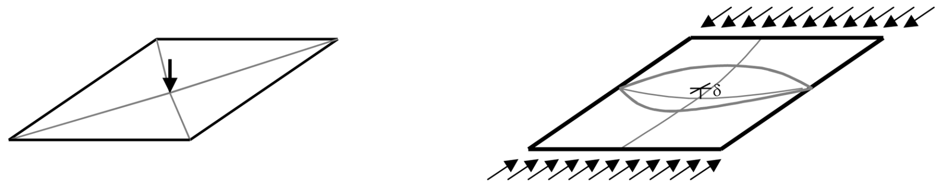


Figure 2. Mechanism models for a simple channel.

work has been undergone to examine optimal yield-line patterns and refine the elements employed (e.g., Askes *et al*., 1999; Gohnert, 2000; Islam *et al*., 1994; Liu, 1999; Ramsay and Johnson, 1997; Rasmussen and Baker, 1998; Thavalingam, 1999). However, classical yield-line analysis has no direct application to cold-formed steel members and generalized yield-line analysis is the focus of the work presented here.

3. Generalized Yield-line Analysis

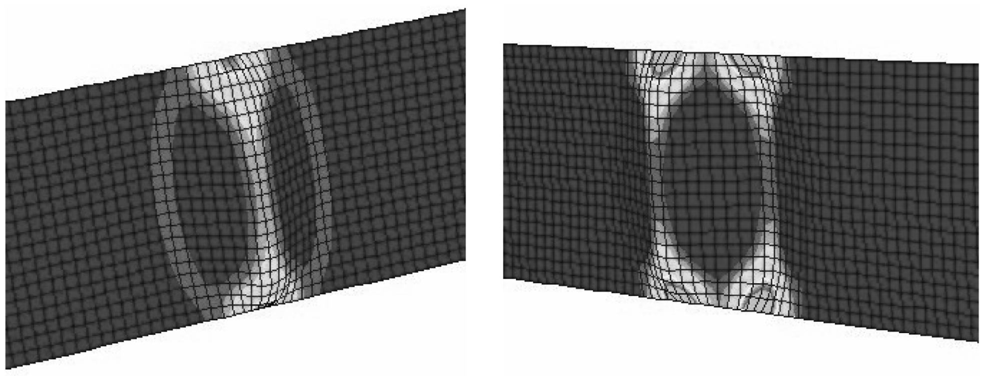
One of the significant challenges for generalized yield-line analysis is that in order to determine the post-buckling load-deformation behavior, the yield-line pattern for the collapse mechanism must be specified *a priori*. The selected yield-line pattern has to reflect the actual developed mechanism as closely as possible to produce meaningful results. Existing research has attempted to determine the exact relationship of the yield-line pattern with respect to material, geometry, boundary conditions and loading. In this vein the work of Murray (Murray and Khoo, 1981; Murray, 1984) has been the most influential. Murray examined the failure patterns in tested members and concluded that all mechanisms can be considered as a sum of simpler basic mechanisms which fit together with compatible deflections



(a) classical yield-line analysis of a simply supported plate (slab) with out-of plane load, yield-lines and patterns develop from *first-order* forces and moments

(b) generalized yield-line analysis of a slender simply supported plate with far-field applied in-plane load, yield-lines and patterns develop from consideration of *second-order* forces and moments

Figure 3. Prototypical examples of classic and generalized yield-line analysis.



(a) roof mechanism (b) flip-disk mechanism

Figure 4. Failure mechanisms at mid-length in nonlinear FE models of simply supported flat plates under in-plane compressive load, plastic strain is shown as contours on deformed shape.

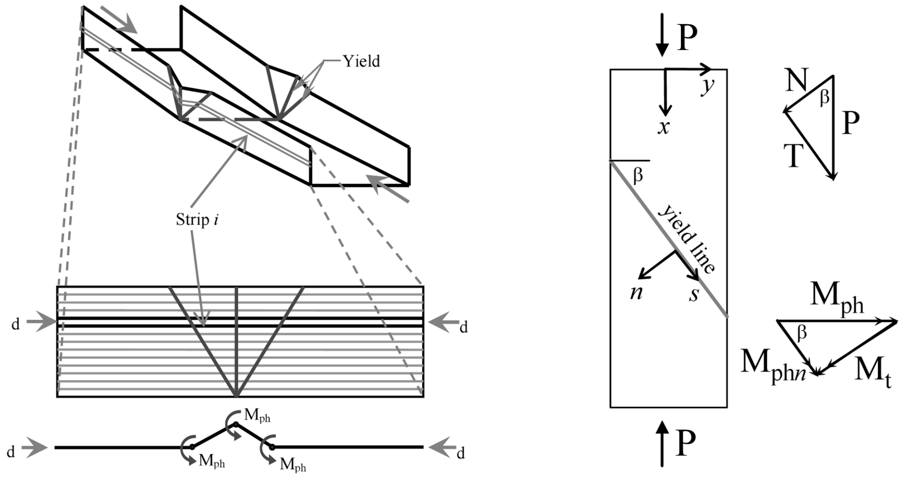
to form the whole mechanism. Murray provided load-deformation relationships for his basic mechanisms. Extensions to Murray's mechanisms are discussed in Zhao (2003).

For example, Murray observed that a plate under in-plane compression has two dominant mechanisms, termed "roof" and "flip-disk" as shown in Fig. 4. Mahendran and Murrary (1991) and Mahendaran (1997) investigated these two mechanisms in some detail and postulated that, for a given yield stress, which of the two mechanisms occurs depends on the initial imperfection magnitude and plate width, both normalized with respect to thickness. Through FE analysis conducted by the authors, as shown in Fig. 4, it was found that the imperfection *shape* is as important as imperfection *magnitude* in determining the resulting mechanism. Further, the boundary between the two mechanisms is not always as distinct as Murray's yield-line models would suggest. In addition, relatively large zones of yielded material are possible. Murray's terms mechanisms which require in-plane yielding as "quasi" mechanisms, as opposed to "true" mechanisms which only deform about the yield-line. All mechanisms demonstrate some distributed inelasticity and are thus quasi mechanisms, general guidelines for when a true mechanism can provide an appropriate approximation are unknown. Finally, residual stresses can complicate both the development and interpretation of the yield-lines. See, for example, the developed plasticity in the simple models of Schafer and Peköz (1998) when residual stresses are included.

In addition to the complicated role of imperfections and residual stresses in triggering a given mechanism, determining the most rational mechanism is a significant challenge. Admissible spatial mechanisms can be found that exhibit significantly lower strength than experimentally observed spatial mechanisms. This observation precludes optimization methods that search for the most critical spatial mechanism (as done in computational implementations of classical yield-line analysis) and is a serious impediment to creating robust techniques focused on yield-line analysis. Thus, it is challenging to find simple relationships between geometry and the developed mechanisms; and experimental evidence still remains the most influential means for determining the appropriate mechanism shape.

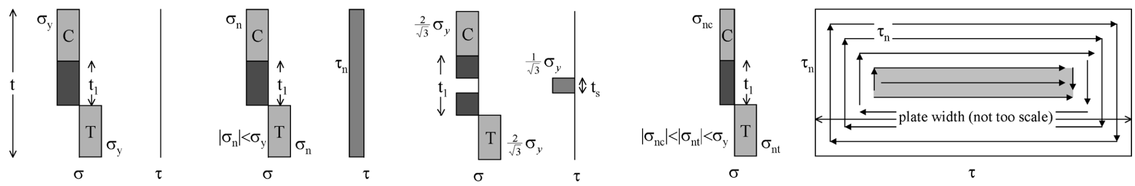
As implemented, generalized yield-line analysis follows either a work or an equilibrium approach. Unlike classical yield-line analysis where the two approaches yield the same result (Jones and Wood, 1967) in generalized yield-line analysis the two methods typically result in different yield surfaces, and thus provide different results (Bakker, 1990; Zhao, 2003). The work method equates the energy dissipated in plastic flow in the yield-lines to the rate of work performed by the external loads to develop the load-deformation relation. The work method was developed by Dean (1976) and has been used by Out (1985) and Bakker (1990).

In the equilibrium approach, the member is divided



(a) yield-line analysis of a channel (b) single inclined yield-line

Figure 5. Conceptual example for equilibrium approach to generalized yield-line analysis.



(a) Murray (1973) (b) Davies (1975) (c) Bakker (1990) (d) Zhao and Hancock (1993)

Figure 6. Fully plastic stress distributions for an inclined yield-line.

into longitudinal strips along the loading direction. Solving equilibrium equations for each strip and then summing across the cross section (Fig. 5a) results in the corresponding load-deformation mechanism curve. The yield-lines are considered piecewise linear across all the individual strips and the equilibrium equations are based on plastic moments developing at these locations. The equilibrium approach is the focus of the work presented here.

4. Inclined Yield-lines

Yield-lines inclined to the direction of load (herein called inclined yield-lines) are of special interest as the plastic moment capacities (M_{ph}) in this case are affected by the action of membrane forces, shear forces, and twisting moments, etc., as shown in Fig. 5b. Further, the spatial collapse mechanisms of cold-formed steel members result in numerous inclined yield-lines (e.g., see Fig. 5a). Thus, the predicted response is dependent on the developed expressions for inclined yield-lines.

Consider a single inclined yield-line (at angle β) with applied in-plane axial force P (Fig. 5b). The axial load (P) creates a first-order force (N) normal to the inclined yield-line and shear force (T) along the inclined yield-line. As deformations proceed, second-order P-δ actions are equilibrated by bending. Bending about the yield-line (M_{phn}) insures a twisting moment (M_t) and the quantity of interest, the moment about the axis perpendicular to the applied load: M_{ph}. M_{ph} is integrated across the member to determine the yield-line's contribution to the collapse response.

A number of different researchers have developed formula for the plastic moment capacity (M_{ph}) of inclined yield-lines (Murray, 1973; Davies *et al*., 1975; Mouty, 1976; Murray, 1984; Bakker, 1990; Zhao and Hancock, 1993; Möller, 1997; Cao *et al*., 1998; Rhodes, 2002) and the developed expressions vary widely. Central to the M_{ph} derivation is the stress distribution at the fully plastic condition, researchers have a pronounced disagreement on this point (Fig. 6). The key differences in the assumptions of the fully plastic stress distribution are (1) the assumed shear stress distribution and (2) the extent to which a yield criterion (von Mises, or Tresca being the most popular) is enforced. All of the models account for the axial load by assuming a central core (of thickness t_1) is yielded in compression. Only Zhao and Hancock (1993) attempt to explicitly include M_t in their model. We may reject Murray's (1973) model of Fig. 6(a), at least on theoretical grounds, on the basis that equilibrium insures the shear (τ) is non-zero, and the stress at yield should not be σ_y, since the stress state is multiaxial not uniaxial.

Figure 7 and Fig. 8 provide a comparison of existing predictions for inclined yield-lines. The plastic bending capacity, M_{ph}, of the inclined yield-line is normalized by:

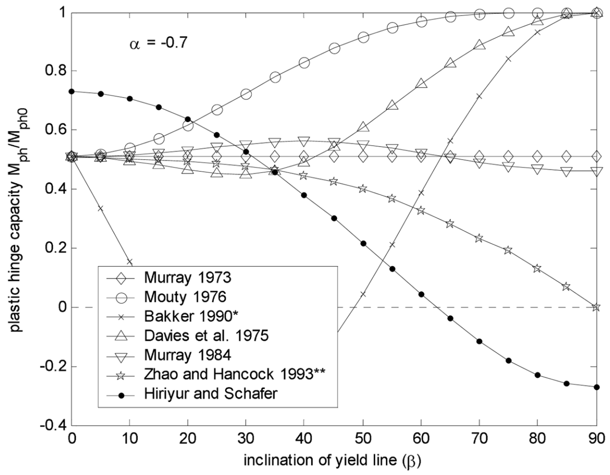


Figure 7. Plastic hinge capacity as a function of the angle of inclination of the yield-line (β) under an applied compression of 70% of the squash load ($\alpha = -0.7$). (Figure motivated by comparison in Cao *et al*. 1998, *as reported in Zhao and Hancock 1993, **von Mises criteria with iteration).

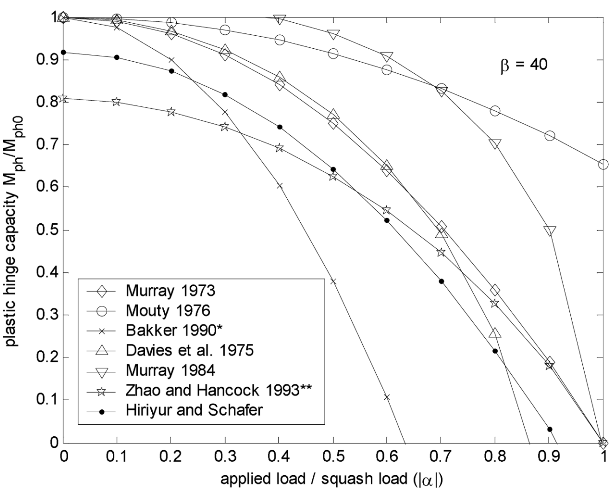


Figure 8. Plastic hinge capacity as a function of the applied compression load for a yield-line inclined 40 degrees (β = 40). (*as reported in Zhao and Hancock 1993, **von Mises criteria with iteration).

$$M_{ph0} = \frac{\sigma_y bt^2}{4}, \tag{1}$$

which for yield stress, σ_y, width, b, and thickness, t, is the plastic bending capacity of a yield-line with no inclination and assuming a one-dimensional state of stress (i.e., ignoring contribution of shear and transverse stresses). The results are also dependent on the level of applied compression P, which is typically normalized by the squash load ($\sigma_y bt$) and expressed in terms of the nondimensional variable, α:

$$\alpha = P/\sigma_y bt \tag{2}$$

As Fig. 7 and Fig. 8 demonstrate, little, if any, consensus exists on this most fundamental building block for generalized yield-line analysis, M_{ph}. As a function of inclination angle, β, existing methods do not converge to similar limit values (at $\beta = 0^o$ and 90^o) and have wildly different functional forms. For a constant inclination angle under increasing application of compressive load the form of the expressions are similar, but the limits are again in disagreement. For many of the earlier methods no strict yield criterion was employed, but if a three-dimensional yield criterion is agreed upon (e.g., von Mises) it is initially hard to understand how such different solutions can emerge.

Murray's models (1973, 1984), which are the simplest, and the most obviously flawed: little or no variation with β, no consideration of shear, or the actual multiaxial stress state at yield; has by far seen the most application. Murray's model has been used in the study of stiffened plates (Murray, 1975), imperfection sensitivity due to mechanism switching (Mahendran and Murray 1991, Mahendran, 1997), collapse of plain thin-walled steel channels (Murray and Khoo, 1981; Dubina and Ungureanu, 2000), web crippling of thin-walled member's at supports and load points (Hofmeyer, 2000; Hofmeyer *et al*., 2000), and moment-curvature predictions for bolted end plates joining rectangular hollow sections (Wheeler *et al*., 1998). In all cases Murray's simple model was able to capture important phenomenological characteristics of the deformation, but the underlying mechanics remains lacking.

5. Proposed Approach for the Stress Distribution

The present theory builds upon the model originally proposed by Bakker (1990). In the existing models, the normal stress along the yield-lines perpendicular to load is assumed to reach the yield stress in a fully plastic state. Bakker shows that if a consistent yield criterion (e.g. von Mises) is used, the normal stress would be different from the yield stress. The von Mises yield function can be written as

$$\psi = \sigma_{nn}^2 + \sigma_{ss}^2 - \sigma_{nn}\sigma_{ss} + 3\sigma_{ns}^2 - \sigma_y^2 \tag{3}$$

where subscript 'n' denotes normal and 's' transverse (see Fig. 5b). By applying the normality condition to the yield function and taking into account the fact that the strain rate ε_{ss} goes to zero, we find that the Poisson's ratio in a fully plastic state reaches its maximum value of 0.5 (incompressible).

$$\varepsilon_{ss} = \nu \cdot \frac{\partial \psi}{\partial \sigma_{ss}} = \nu(2\sigma_{ss} - \sigma_{nn}) = 0 \Rightarrow \frac{\sigma_{nn}}{\sigma_{ss}} = 0.5 \tag{4}$$

This results in normal and transverse stresses satisfying von Mises yield criterion in the fully plastic state:

$$\sigma_{nn} = \frac{2\sigma_y}{\sqrt{3}} \tag{5}$$

$$\sigma_{ss} = \frac{\sigma_y}{\sqrt{3}} \tag{6}$$

With this fully plastic stress distribution, the plastic moment capacity of a normal yield-line is:

$$M^*_{ph0} = \frac{\sigma_y b t^2}{2\sqrt{3}} \tag{7}$$

M^*_{ph0} is 1.15 times greater than M_{ph0} of Eq. (1) which assumes a one-dimensional state of stress at yield.

A simple postulate, verified in our FE modeling, is used for inclined yield-lines: *the principal axes in a fully plastic state are oriented along the yield-lines*. For the plate with a yield-line inclined at angle β, the principal stresses σ_1 and σ_2 are thus σ_{nn} and σ_{ss} of Eq. (5) and (6) above. Transforming these stresses back to the coordinate axis of the strip itself (Fig. 5b) results in:

$$\sigma_{xx} = \frac{\sigma_1+\sigma_2}{2} + \frac{\sigma_1-\sigma_2}{2}\cos 2\beta = \left(\frac{\sqrt{3}}{2} + \frac{\cos 2\beta}{2\sqrt{3}}\right)\sigma_y \tag{8}$$

$$\sigma_{yy} = \frac{\sigma_1+\sigma_2}{2} - \frac{\sigma_1-\sigma_2}{2}\cos 2\beta = \left(\frac{\sqrt{3}}{2} - \frac{\cos 2\beta}{2\sqrt{3}}\right)\sigma_y \tag{9}$$

$$\sigma_{xy} = \frac{\sigma_1-\sigma_2}{2}\sin 2\beta = \left(-\frac{\sin 2\beta}{2\sqrt{3}}\right)\sigma_y \tag{10}$$

ABAQUS models of a thin plate with a single inclined yield-line were analyzed and the stress at the seven integration points through the thickness were recorded and compared to the predictions in Eq.'s (8)-(10). The model employed shear flexible shell elements and isolated the inelastic (elastic-perfectly plastic) behavior in a single inclined line of elements. The results are plotted in Fig. 9 for the model in a fully yielded state and show exact correlation between the predicted stress distribution and that observed in the model in ABAQUS when von Mises yield criterion is enforced in a localized inclined yield-line.

From Eq. (8) the stress σ_{xx} in the yield-line is factored from the yield stress (σ_y) by a function λ, which depends on the angle of inclination of the yield-line, where λ is:

$$\lambda = \frac{\sqrt{3}}{2} + \frac{\cos 2\beta}{2\sqrt{3}} \tag{11}$$

Knowing σ_{xx} and recognizing that a yielded core at the center of thickness, t_1, does not contribute to the bending strength, the moment capacity of the inclined yield-line per unit width is:

$$M_{ph} = \frac{\lambda\sigma_y}{4}(t^2 - t_1^2) \tag{12}$$

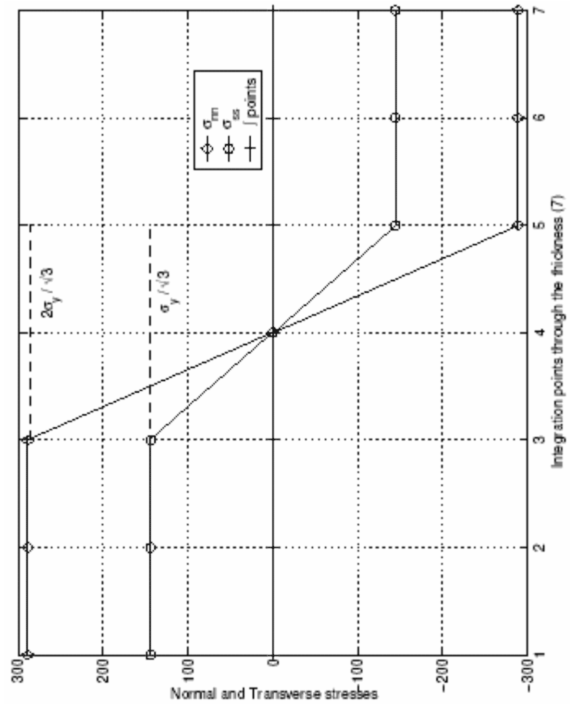


Figure 9. Normal and transverse stress distribution through the thikness of a yield-line as predicted by an FE model in ABAQUS with von Mises yield criterion.

The core, at stress σ_{xx}, carries the axial load over thickness t_1, using Eq. (2) we find

$$\lambda\sigma_y b t_1 = P = \alpha\sigma_y b t \Rightarrow t_1 = \alpha t/\lambda \tag{13}$$

Substituting Eq. (13) into (12) results in the moment capacity of an inclined yield-line per unit width:

$$M_{ph} = \frac{\lambda\sigma_y t^2}{4}\left(1 - \left(\frac{\alpha}{\lambda}\right)^2\right) \tag{14}$$

Finally, the ratio of the moment capacities of an inclined yield-line to a straight yield-line is given by

$$\frac{M_{ph}}{M^*_{ph0}} = \frac{\sqrt{3}\lambda}{2}\left(1 - \left(\frac{\alpha}{\lambda}\right)^2\right) \tag{15}$$

The developed model for an inclined yield-line (Eq. 14) is normalized by the solution for a straight yield-line assuming a one-dimensional state of stress (Eq. 1, M_{ph}/M_{ph0}) and plotted in Fig. 7 and Fig. 8.

Considering the case of an applied load of 70% of the squash load (α = 0.7) and varying yield-line inclination (Fig. 7) the proposed method has two significant differences with existing work: (1) the predicted strength at no inclination (β = 0) is greater than the other models, (2) negative hinge capacity is predicted for large inclination angles. The first difference is due to the fully plastic stress state, specifically that the yielded central core t_1 is $\alpha t/\lambda$ instead of αt as assumed in all other models. The second difference indicates that no bending capacity exists for yield-lines inclined greater than 60% (for $\alpha = 0.7$). For larger inclination angles bending capacity must actually come from other parts of the member, or the mechanism will be unstable. As the yield-line inclination angle (β) approaches parallel (90^o) to the applied load the other methods force convergence to either full, or no capacity (as is shown for all but Murray's models) since the situation is physically unrealizable it is unclear why either specific limit is appropriate.

For a yield-line with a constant inclination of 40^o under increasing load (Fig. 8) the proposed model and Zhao and Hancock (1993) predict a reduction in M_{ph} even with an infinitesimal applied load (α~0). This reduction is due to the stress state and consistent use of von Mises yield criteria. The majority of existing models begin with a plastic hinge capacity of M_{ph0} when no load is applied, and no capacity when the full squash load is applied. The proposed model and Bakker's model both predict loss of the hinge capacity before reaching the full squash load ($\alpha = 1$), again this is due to the stress state. At other inclination angles the conclusions drawn may differ, for low β ($<30^o$) the proposed model gives greater capacity than the majority of models, but for high β the capacity is less; essentially following the trends established in Fig. 7.

6. Implementation

A program capable of computing the load-displacement (P-Δ) response for a given cold-formed steel member with a specified spatial collapse mechanism was created in MATLAB. The first step in the computation is to generate the displacement field for the specified yield-line pattern (collapse mechanism). The displacement field is calculated as the first buckling mode of a developed FE model of the plate under an in-plane reference load, where the bending capacity along the yield-lines has been set artificially low. The FE model employs a triangular plate element mesh with duplicate nodes introduced along the yield-lines and connected together with hinges as shown in Fig. 10. This condition enables the structure to rotate freely along the specified yield-lines while maintaining translational compatibility. The stiffness matrix formulations for the triangular plate elements follow Batoz *et al* (1980). Additional information on the developed program is available in Hiriyur (2003).

Once the displacement field is generated, the structure is divided into longitudinal strips and equilibrium equations developed using the plastic moment capacities. Along each longitudinal strip, one rigid element is selected for which the equilibrium equation is formulated to solve for the load capacity. Based on the inclination of the yield-lines, the plastic moments at the ends of the element corresponding to strip i are calculated as M_{iy1} and M_{iy2} using equation (14). The equilibrium equation thus becomes:

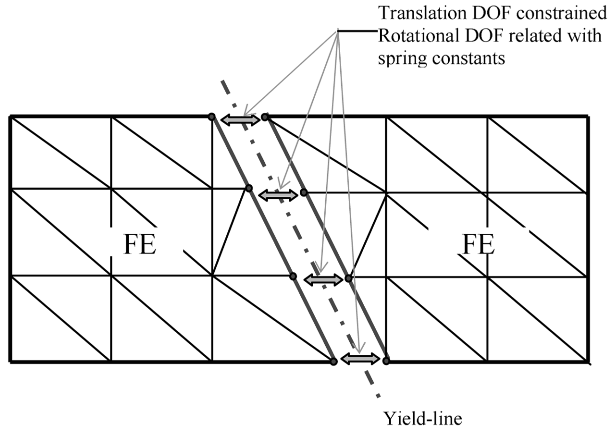


Figure 10. FE mesh of rectangular plate with yield-line constraints.

$$M_{iy1} + M_{iy2} - P_i\Delta_i = 0 \tag{16}$$

In Eq. (16) both M_{iy1} and M_{iy2} are functions of the axial load P_i and Eq. (16) represents a nonlinear curve of P_i vs. Δ, for strip i. For the range of Δ specified, the above equation is solved for P_i using the bisection method. The values of P_i are summed for all strips to generate the member P-Δ curve.

7. Applications

7.1. Yield-line normal to load

The proposed method is examined first on the simplest case of a rectangular plate with one straight yield-line perpendicular to the loading direction. The loading and support edges are simply supported and the side edges are free. The equation for the plastic mechanism curve can be derived analytically for this case.

Since the yield-line is perpendicular to the loading direction the stress factor λ is

$$\lambda = \frac{\sqrt{3}}{2} + \frac{1}{2\sqrt{3}} = 1.155 \tag{17}$$

Denoting the thickness of the central core as t_1, the axial load resistance and the plastic moment are

$$P = \lambda\sigma_y b t_1 \Rightarrow t_1 = \frac{P}{\lambda\sigma_y b} \tag{18}$$

$$M = P\Delta = \frac{\lambda\sigma_y b}{4}(t^2 - t_1^2) \tag{19}$$

The above equations can be rearranged to form a quadratic relationship, which can be solved for P in terms of the out of plane displacement Δ:

$$P = \lambda\sigma_y b(\sqrt{4\Delta^2 + t^2} - 2\Delta) \tag{20}$$

A rectangular plate with a single straight yield-line

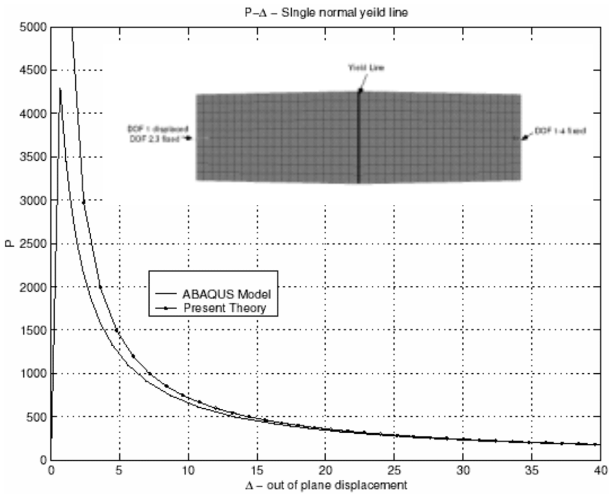


Figure 11. Normal yield-line P-**Δ** comparison (inset: ABAQUS model).

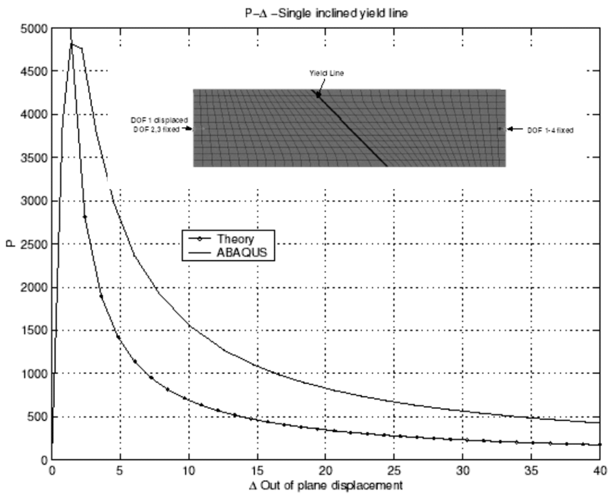


Figure 12. Inclined yield-line P-**Δ** comparison (inset: ABAQUS model).

perpendicular to the in-plane loading direction is modeled using ABAQUS. Four node shear flexible shell elements (S4R) are used in the developed FE model. The central yield-line portion is assigned elastic-perfectly plastic material properties while the rest of the plate is assigned purely elastic material properties. Second order nonlinear displacement controlled FE analysis was performed on this member to generate the load-deformation curves. Therefore, the "load" was applied by displacing one ends towards the other, as shown in the insets of Fig. 11 through Fig. 13. Boundary conditions are also given in the insets, with the 1 direction along the length of the plate, the 2 direction transverse, and the 3 direction perpendicular to the plate. A small initial imperfection is introduced to engage out-of-plane deformations. Seven integration points were assigned through the thickness. The normal, transverse and shear stress magnitudes at the seven integration points along the thickness of the yield-line elements in the fully plastic state are presented in Fig. 9. It may be observed that the stresses correspond exactly to the predicted values. The load-deformation curve obtained from ABAQUS is compared (Fig. 11) against predicted curve and shows reasonable agreement.

7.2. Inclined yield-line

An ABAQUS model of a rectangular plate with single yield-line inclined at an angle $\beta = 45^o$ with the in-plane loading direction is produced. As in the previous case, a second order nonlinear finite element analysis was performed using four node quadrilateral shell elements. While the stress distribution along the thickness was again found to be in perfect correlation, the load deformation curves (Fig. 12) are not entirely in agreement. This disagreement is examined further in the final example.

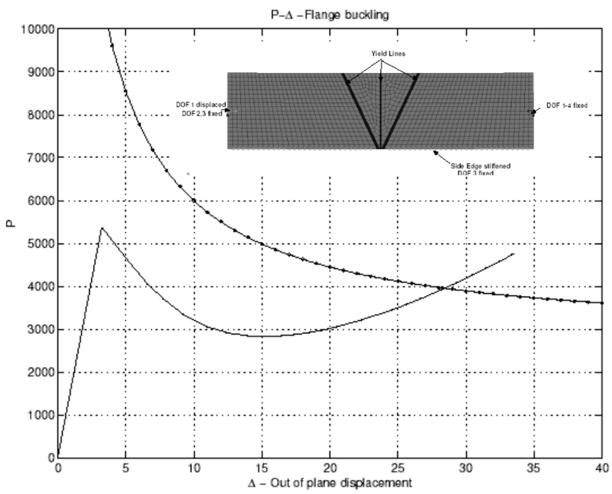


Figure 13. Plate stiffened on one edge, P-**Δ** comparison (inset: ABAQUS model).

7.3. Yield-lines in an unstiffened plate

A similar application to the previous two analyses is carried out for the case of a plate with multiple yield-lines. In the ABAQUS model, initial imperfections were enforced using the first buckling mode and displacement controlled nonlinear finite element analysis was performed. For the yield-line method, the plate is divided into longitudinal strips and equilibrium equations formulated for each displaced strip element. The equations are solved and summed to generate the plate load-deformation curve, which is compared to the corresponding curve obtained using ABAQUS (Fig. 13).

8. Discussion

The present model subdivides the member into strips and adopts an equilibrium approach to generate the $P - \Delta$

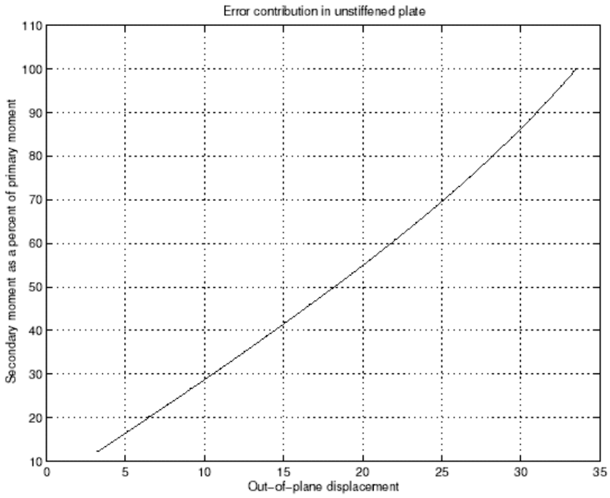


Figure 14. Development of secondary moment in the unstiffened element.

curves. Only a first-order equilibrium, taking into account the effect of moments generated by out-of-plane displacement, is considered. This would be a reasonable assumption for many cases where the out-of-plane degree of freedom dominates. In such cases, the load deformation relationship predicted from the present theory matches well with the finite element model; however for inclined yield-lines significant error is observed.

For the case of an inclined yield-line, the stress distributions assumed by the theory match well with the ABAQUS results. However, the stress contour of the von Mises stress outside of the yield-line and in the main FE model indicate additional stresses induced by twisting. The mechanism that forms does not remain straight and simple summation of the longitudinal strips ignores the transverse movement. These effects are not included in the present analysis and need to be addressed in the future. For the unstiffened plate corresponding to the face of an angle with three yield-lines, the $P-\Delta$ curve generated using the present theoretical model has little correlation with the corresponding ABAQUS curve. This can be explained by observing the displacement field generated by the particular collapse mechanism. The transverse degree of freedom is nearly as large as the out-of-plane displacement, resulting in the development of secondary moments about the out-of plane axis. Since the effect of these moments is not included in the present model, the difference in the load capacities is observed. Figure 14 shows the increasing effect of secondary moment as the analysis progresses, even as the mechanism first forms (D~5) the error is at least 15% and increases quickly.

The treatment for the state of stress in a yield-line (Eq. 8-10) is verified by the FE models, but the summation of these stresses using a conventional equilibrium approach does not provide a solution with the desired accuracy. The full kinematics of the developed mechanism must be considered in order to accurately develop the mechanism curve. The authors recommend that the work (energy) approach to development of the mechanism curve, with the stresses derived herein, may provide a more robust and convenient solution methodology and intend to pursue efforts in that direction in the future.

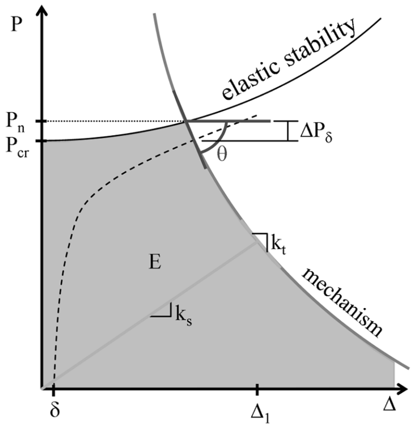
Nonlinear finite element solutions employing shell elements, von Mises yield criteria, and large displacement effects are the focus of the validation efforts presented here, because the developed method is intended to be a direct simplification of the more general FE analysis. Additional validation may be performed by experimental testing. For example, Zhao and Hancock (1993b) performed tests on steel plates with specially stiffened cross-sections to force yielding at defined locations. Zhao (2003) summarizes additional tests that have been conducted, including those on both open and closed sections. Comparison of the general FE solutions, as well as an improved version of the simplified yield-line analysis as discussed here, is an important topic for future study.

8.1. Potential applications for generalized yield-line analysis

While significant work remains on developing a generally applicable yield-line analysis method for cold-formed steel members, that is applicable in the large deformation range, and based on fundamental mechanics, the need for a robustly developed method remains. This section discusses the potential for generalized yield-line analysis and suggests ways in which the developed method could be used to improve our understanding of the behavior, and our ability to design, cold-formed steel members.

8.2. Imperfection sensitivity

Cold-formed steel members are imperfection sensitive, a traditional cause of complication and uncertainty in design. By understanding the role of failure mechanisms in creating imperfection sensitivity, generalized yield-line analysis can provide unique insight on this long-standing problem. Formal evaluations of imperfection sensitivity typically follow Koiter (1967) and use perturbation techniques about the lowest eigenmode to determine higher order characteristics of the related equilibrium path, where negative slopes indicate unstable imperfection sensitive structures (e.g., Pignataro 1996, Rondal 1998). Such an approach relies on elastic stability (although large deformation) and *ignores* plastic deformation. While elastic stability measures of imperfection sensitivity are appropriate in some cases, observable imperfection sensitivity (i.e., large variation in maximum capacity, ΔP_δ of Fig. 15, for nominally identical specimens) may result even with stable elastic post-buckling branches if a mechanism cuts-off the capacity before the influence of imperfections has been "damped" out, as illustrated in Fig. 15. Imperfection sensitivity of this type has been



Application of mechanism curve

P	proportional to applied load
Δ	prop. to out-of-plane displacement
P_{cr}	elastic buckling load (path from P_{cr} is the elastic post-buckling path & may be approximated by perturbation)
P_n	upperbound estimate of ult. strength
δ	prop. to out-of-plane imperfection
ΔP_δ	Estimate of the capacity loss due to imperfection of magnitude δ
θ	angle of initial collapse, candidate for imperfection sensitivity metric
E	energy due to collapse (shaded area)
k_t	tangent stiffness at damage (defl'n) Δ_1
k_s	secant stiffness at damage (defl'n) Δ_1

Figure 15. Potential applications for mechanism curve developed from generalized yield-line analysis.

demonstrated numerically in thin-walled cold-formed steel beams (Schafer 1997) and is generally the source of variation in strength observed in cold-formed steel models. It is postulated, that in this case, the slope (θ) of the mechanism that intersects the elastic branch governs the degree of imperfection sensitivity. This is based on the finding that the angle of the elastic post-buckling branch governs imperfection sensitivity for members with unstable elastic post-buckling response (Bazant 1991). The use of θ as an imperfection sensitivity metric deserves further investigation.

8.3. Ultimate strength in design

The intersection of the elastic post-buckling response and the mechanism curve (P_n of Fig. 15) provides an upperbound estimate of the ultimate member strength. Such an intersection requires estimation of the elastic post-buckling branch as well as the mechanism curve, but is nevertheless quite promising if the approach could be efficiently generalized. Relying on Murray's work for defining the mechanism Ungurenau and Dubina (2004) have provided a related design method in which yield-line analysis can be directly incorporated into strength prediction, including interactions. The challenges in successfully implementing ultimate strength design methods that rely on yield-line analysis are (1) developing appropriate spatial mechanisms, (2) insuring the generalized yield-line analysis applied is true to the mechanics involved and (3) providing methods for applying the approach to different cross-sections. Despite these challenges the work is sorely needed.

8.4. Ductility, energy absorption, and performance-based design

The design of cold-formed steel members for dynamic loads: seismic, blast, etc., can be significantly enhanced with the application of generalized yield-line analysis. As Fig. 15 demonstrates, the elastic post-buckling response, combined with generalized yield-line analysis can estimate: (1) stiffness at any damage state, (2) energy absorbed up to any damaged state, and (3) ultimate ductility (if combined with other limiting criteria). The ability to provide this information in an analytical form without recourse to general purpose FE analysis is a significant benefit. The current push towards performance-based design of structures makes generalized yield-line analysis a particularly attractive tool. Conventional civil/structural design is only concerned with service load behavior (typically the elastic stiffness) and ultimate strength. For performance-based design to be effective, predictions of the strength must be provided for any level of damage. Cold-formed steel members, e.g., cladding, purlins, girts, and joists are often secondary members in civil/structural applications, it may be expected that such secondary members will experience significant damage even when a building is only under globally moderate demands. Therefore, understanding the full system (building) response even under moderate demands requires estimation of post-peak strength, and stiffness for numerous cold-formed steel members.

8.5. Competitive buckling modes and triggered mechanisms

While a great deal of study has been conducted on understanding elastic buckling modes and the ramifications of closely spaced buckling modes (compound buckling, imperfection sensitivity, etc.) little has been done to understand the role of different, competing, failure mechanisms. It is generally assumed that buckling modes and failure mechanisms are related in thin-walled structures. If a plate may fail in competing mechanisms a loss in strength may be observed when the mechanism switches. For example, nominally identical flat plates in compression undergoing local buckling experience a 5% drop in peak load as the collapse mechanism switches from a "roof" to a "flip-disc" mechanism (Mahendran 1997). For the common cold-formed steel lipped channel both the buckling modes and mechanisms are in competition and the loss in capacity may not be so benign. In fact, failures associated with distortional buckling have been observed to have a strongly reduced post-buckling capacity and greater imperfection sensitivity than local buckling. (Hancock *et al*. 1994, Schafer and Peköz 1999 and Schafer 2002). The mechanism associated with distortional buckling appears to cut off the post-buckling branch for local buckling, but little is known about this mechanism and how it is triggered.

9. Conclusions

Generalized yield-line analysis, which provides an approximation of the complete load-deformation inelastic collapse response of a cold-formed steel member under in-plane load, is distinct from classical yield-line analysis as typically employed in the design of transversally loaded concrete slabs, or steel connections. A number of deterrents exist to the widespread use of generalized yield-line analysis: (1) the method requires *a priori* determination of the spatial mechanism of interest, (2) results are sensitive to the selected mechanism, (3) no automated means exist for selecting appropriate mechanisms, and (4) significant disagreement exists on the fundamental quantities underlying the method. Spatial collapse mechanisms typically include yield-lines inclined from the loading direction; but development of appropriate expressions for this case have lead to widely varying solutions. Based on the von Mises yield criterion, in the fully plastic state ($\nu = 0.5$), and assuming only normal and transverse stresses exist for bending about the inclined yield-line axis, new expressions are provided for the bending strength of an inclined yield-line. The assumed stress state is verified by finite element analysis; however, it is found that development of the correct load-deformation response requires further information than just the correct local plastic stresses. The kinematics of the actual deformation, including both out-of-plane and transverse moments must be faithfully re-produced, a significant challenge within the traditional equilibrium method of generalized yield-line analysis. Adoption of a work approach is one possible avenue for future work. Finally, despite the challenges and deterrents to developing generalized yield-line analysis the advantages of such a method are significant. Specific applications for generalized yield-line analysis in the study of imperfection sensitivity, ultimate strength prediction, ductility and energy absorption predictions, and the study of competing failure mechanisms are suggested.

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