Appendices

Appendix 1. Virtual Work Principle

In this section, we provide a formal proof of the virtual work principle. We will make use of the notations presented in Fig. 3.2a. First, let us recognize the fact that in the solid mechanics problems, there are two different sets involved:
- Kinematic variables (deformation related): \( \mathbf{u}, \mathbf{\varepsilon} \)
- Static variables (load related): \( \mathbf{\sigma}, \mathbf{t}, \mathbf{b} \)

If these quantities are actual solution of a problem, then \( \mathbf{\sigma} \) and \( \mathbf{b} \) will be related by the equilibrium equations:

\[
\mathbf{\sigma}_{ij,j} + b_i = 0 \quad \text{in} \quad \Omega \tag{A1.1}
\]

and the stress will be related to the strain by a relationship such as (not necessarily an elastic relation)

\[
\mathbf{\sigma}_{ij} = g(\mathbf{\varepsilon}_{ij}) \tag{A1.2}
\]

and to the boundary traction by

\[
t_i = \mathbf{\sigma}_{ij} n_j \tag{A1.3}
\]

We are often interested in approximate solutions, and in the process of seeking an approximate solution, we start with functions that may not satisfy all of the relations exactly and simultaneously. In this context, let us define a special set of static and kinematic variables as follows:

- Kinematically admissible set \( (\mathbf{u}, \mathbf{\varepsilon}) \):
  - (a) One where \( \mathbf{u} \) and \( \mathbf{\varepsilon} \) are related to each other by the strain-displacement relation
    \[
    \mathbf{\varepsilon}_{ij} = \frac{1}{2} \left( \mathbf{u}_{ij,j} + \mathbf{u}_{ji,j} \right) \tag{A1.4}
    \]

- Statically admissible set \( (\mathbf{\sigma}, \mathbf{t}, \mathbf{b}) \):
  - One which satisfies the equilibrium equation as
    \[
    \mathbf{\sigma}_{ij,i} + b_i = 0 \quad \text{in} \quad \Omega \quad \text{and the } \left( \mathbf{\sigma}, \mathbf{t} \right) \text{ relation}
    \]
    \[
    t_i = \mathbf{\sigma}_{ij} n_j \tag{A1.5}
    \]

Again, it is reminded that the two sets \( (\mathbf{u}, \mathbf{\varepsilon}) \) and \( (\mathbf{\sigma}, \mathbf{t}, \mathbf{b}) \) are not necessarily related at this point. The displacement \( \mathbf{u} \) is referred to as the virtual (not actual) displacement. We will now prove the virtual work principle starting with the external virtual work

\[
\mathbf{W}^e = \int_{\Gamma} t_i \mathbf{u}_i ds + \int_{\Omega} b_j \mathbf{u}_j dv \tag{A1.7}
\]

\[
= \int_{\Gamma} \mathbf{\sigma}_{ij} n_j \mathbf{u}_i ds + \int_{\Omega} b_j \mathbf{u}_j dv \quad \text{(from Eq. A1.6)}
\]
\[ \begin{align*}
&= \int \left( \tilde{\sigma}_y \tilde{u}_j \right)_j \, ds + \int b_i \tilde{u}_i \, dv \text{ (using Gauss theorem) } \\
&= \int (\tilde{\sigma}_{y,j} + b_j \tilde{u}_j) \, dv + \int \tilde{\sigma}_y \tilde{u}_{i,j} \, dv \\
&= 0 + \int \tilde{\sigma}_y \tilde{u}_{i,j} \, dv \text{ (from Eq. A1.5)} \\
&= \int \tilde{\sigma}_{ij} \left[ \frac{1}{2} (\tilde{u}_{i,j} + \tilde{u}_{j,i}) + \frac{1}{2} (\tilde{u}_{i,j} - \tilde{u}_{j,i}) \right] \, dv \\
&= \int \tilde{\sigma}_{ij} (\bar{\epsilon}_{ij} + \bar{\omega}_{ij}) \, dv
\end{align*} \]

where \( \bar{\omega}_{ij} = \frac{1}{2} \left( \tilde{u}_{i,j} - \tilde{u}_{j,i} \right) \) is the antisymmetric displacement gradient, with the property \( \bar{\omega}_{ij} = -\bar{\omega}_{ji} \) and \( \bar{\omega}_{ij} = 0 \) for \( i = j \). The product of a symmetric tensor like \( \tilde{\sigma}_y \) with an antisymmetric tensor like \( \bar{\omega}_{ij} \) is zero (since the product of the diagonal terms are zero due to the fact that the diagonal elements of \( \bar{\omega}_{ij} \) are zero, and the remaining terms can be grouped and shown to be zero as, for instance, \( \bar{\omega}_{12} \bar{\omega}_{12} + \bar{\omega}_{21} \bar{\omega}_{21} = \bar{\omega}_{12} \bar{\omega}_{12} - \bar{\omega}_{12} \bar{\omega}_{12} = 0 \)).

Thus,
\[ \bar{W} = \int \tilde{\sigma}_y \bar{\epsilon}_y \, dv = \int \bar{U}_0 \, dv = \bar{U} \quad (A1.8) \]

where \( \bar{U} \) is the virtual internal energy and
\[ \bar{U}_0 = \tilde{\sigma}_y \bar{\epsilon}_y \quad (A1.9) \]

is the virtual internal energy density (energy per unit volume). Explicitly, the virtual work principle states:
\[ \int \tilde{t}_i \tilde{u}_i \, ds + \int b_i \tilde{u}_i \, dv = \int \tilde{\sigma}_y \bar{\epsilon}_y \, dv \quad (A1.10) \]

In symbolic notation, Eq. A1.10 is written as Eq. 3.2.

### Appendix 2. General Stress-Strain Relation for an Elastic Material

There is a special relationship that all elastic materials, linear or nonlinear, must obey, and we derive that relationship here with the aid of the laws of thermodynamics. The first law concerns the conservation of energy. Referring to the system shown in Fig. 5.2a, define,

- \( \dot{W} \) = rate of work done by the external forces on the system
- \( \dot{U} \) = rate of internal energy stored
- \( \dot{V} \) = rate of energy stored as kinetic energy
- \( \dot{W}_1 \) = rate of other forms of energy (e.g., chemical, thermal, electromagnetic, etc.) supplied to the system
\[ \dot{W}_2 = \text{rate of other forms of energy (e.g., chemical, thermal, electromagnetic, etc.) withdrawn from the system} \]

The first law of thermodynamics states that the energy cannot be destroyed. Thus, for a purely elastic material, the energy balance reads:

\[ \dot{W}_1 - \dot{W}_2 = \dot{U} + \dot{V} \quad (A2.1) \]

(For an inelastic material, a term reflecting internal energy dissipation must be added to the equation.) Neglecting \( \dot{V} \), \( \dot{W}_1 \), and \( \dot{W}_2 \), one has for an elastic materials:

\[ \dot{W} = \dot{U} \quad (A2.2) \]

Considering an infinitesimal time \( \delta t \), Eq. A2.2 can be written as

\[ \frac{\delta W}{\delta t} = \frac{\partial U}{\partial t} \]

and for a nonzero \( \delta t \),

\[ \delta W = \delta U \quad (A2.3) \]

Now, we make use of the virtual work expression given by Eq. A1.10. Let the variational displacement \( \delta u \) be our virtual displacement to use in Eq. A1.8, where \( u \) is the actual displacement. Let \( \delta u \) be applied over a time period \( \delta t \). Let \( \delta \varepsilon \) the corresponding virtual strain. Let the static admissible set to be the actual stresses and tractions. Then by Eq. A1.8,

\[ \delta W = \delta U = \int_{\Omega} \sigma_{ij} \delta \varepsilon_{ij} \, dv \quad (A2.4) \]

From Eqs. A2.3 and A2.4, one has

\[ \delta U = \int_{\Omega} \sigma_{ij} \delta \varepsilon_{ij} \, dv \quad (A2.5) \]

Eq. A2.5 gives a quantitative equation for calculating the change in internal energy of the system in terms of stresses and strains during an infinitesimal change. By differentiating Eq. A2.5, we obtain the relation that we sought:

\[ \sigma_{ij} = \frac{\partial U}{\partial \varepsilon_{ij}} \quad (A2.6) \]

Eq. A2.6 states that for an elastic material, the stresses can be derived from a scalar function (also called the potential function) – the internal energy function \( U \). For the special case of linear elastic material, \( U \) is a quadratic function of strain as

\[ U = \varepsilon^T \mathbf{D} \varepsilon \quad (A2.7) \]

and from Eq. A2.6, one recovers the generalized Hooke’s law

\[ \sigma = \mathbf{D} \varepsilon \quad (A2.8) \]

**Appendix 3: Computer Coding Guidelines for Algorithm 8.2: Global Iteration Strategy for Model Calibration**

**Definition of variables:**

NGMAX: Maximum number of global iterations permitted
ERMAX: TOL
NPROP: Number of model parameters
NPIV: Number of PIVs
NUM: Number of increments that load is to be divided into
IELAST = 1: Find elastic stiffness, = 0: Find elasto-plastic stiffness
ISTIFF = 1: Find stiffness, = 0: Do not find stiffness
ISTRES = 1: Find incremental stress, = 0: Do not find incremental stress

Definition of arrays:
SIGB(6): \( \sigma_n \)
SIGF(6): \( \sigma_{n+1} \)
SIGSB(6): \( \sigma'_n \)
SIGSF(6): \( \sigma'_{n+1} \)
SIGE(6): \( \sigma^e_{n+1} \)
EPB(6): \( \varepsilon_n \)
DEP(6): \( \Delta \sigma \)
DSIG(6): \( \Delta \varepsilon \)
PROP(NPROP): \( E, \nu, \) etc.
STOR(2*NPIV+1): 1 to NPIV NPIV+1 to 2*NPIV 2*NPIV+1

ICOD(6) = 1 strain control, = 0 for stress control
V(6): Combination of \( \sigma^e \) and \( \varepsilon^e \) depending on ICOD(6)
DV(6): Combination of \( \Delta \sigma^e = \sigma^e / \text{NUM} \) and \( \Delta \varepsilon^e = \varepsilon^e / \text{NUM} \) depending on ICOD(6)
R(6): Temporary array (incremental load)
R0(6): Residual stress
C(6,6): \( D'_{n+1} \)

Input Data
Read in or type in:
NPROP
NPIV
NGMAX
ERMAX
(PROP(I),I=1,NPROP)
(SIGB(I),I=1,6)
(STOR(I),I=1,NPIV)
((ICOD(I),V(I)),I=1,6),NUM
XXX=1.0E+20 (a large value)

Initialize
EPB(I)=0; SIGSB(I)=SIGB(I); SIGE(I)=SIGB(I) for I=1,6
DV(I)=\( [V(I)-SIGB(I)] / \text{NUM} \) if ICOD(I)=0
\( = [V(I)-EPB(I)] / \text{NUM} \) if ICOD(I)=1 for I=1,6
Start Analysis

Start load step loop

FOR INC=1,NUM
Set DEP(I)=0; DSIG(I)=0;
IF[ICOD(I)=0] then SIGE(I)=SIGE(I)+DV(I) for I=1,6

Start iteration loop

FOR ITNO=1,NGMAX
C Call material model module and calculate tangent stiffness
ISTIFF=1; ISTRE=0
IF[ITNO=1] then IELAST=1; else IELAST=0
CALL MISES(C,PROP,STOR,SIGB,EPB,DSIG,DEP,NPROP,NPIV,
IELAST,ISTIFF,ISTRE)

C Modify for strain-controlled loading
FOR I=1,6
IF(ICOD(I)=1) C(I,I)=XXX
IF(ITNO=1) THEN
IF(ICOD(I)=0) R(I)=DV(I)
IF(ICOD(I)=1) R(I)=DV(I)*XXX
ELSE
IF(ICOD(I)=0) R(I)=R0(I)
IF(ICOD(I)=1) R(I)=0
ENDIF
NEXT I

C Solve $\mathbf{D}_{n+1}\delta\mathbf{e} = \delta\mathbf{\sigma}$
CALL SOLVE(C,R)
(R contains stress on entering, and strain on returning)

C Define incremental strain from iterative strain
DEP(I)=DEP(I)+R(I) for I=1,6

C Call material model module and calculate incremental stress
ISTIFF=0; ISTRE=1; IELAST=0
CALL MISES(C,PROP,STOR,SIGB,EPB,DSIG,DEP,NPROP,NPIV,
IELAST,ISTIFF,ISTRE)

C Update stresses and spring forces and define residual load
FOR I=1,6
SIGF(I)=SIGB(I)+DSIG(I)
SIGSF(I)=SIGSB(I)+DSIG(I)
R0(I)=SIGE(I)-SIGSF(I)
NEXT I

C Define error
ERR=0.0
ERRT=0.0
FOR I=1,6
ERRT=ERRT+SIGSF(I)**2
IF(ICOD(I)=0) ERR=ERR+R0(I)**2
NEXT I
ERR=SQRT(ERR)/SQRT(ERRT)
C Check for convergence
   IF(ERR<ERMAX) Exit iteration loop

C End of iteration loop
   NEXT ITNO

C Update strains, stresses, spring forces and PIVs
   FOR I=1,6
      SIGB(I)=SIGF(I)
      SIGSB(I)=SIGSF(I)
      EPB(I)=EPB(I)+DEP(I)
   NEXT I

   STOR(I)=STOR(I+NPIV) for I=1,NPIV
C Print results and continue with the next load step
   NEXT INC

Appendix 4: Convexity of a Function

Fig. A4.1. Schematic of a One-Dimensional Smooth Convex Function

Referring to Fig. A4.1, the function $\phi(\sigma)$ is convex if

$$\bar{\phi} \leq \phi^*$$  \hspace{1cm} (A6.1a)

where

$$\bar{\sigma} = \beta\sigma_1 + (1-\beta)\sigma_2; \hspace{0.5cm} \beta \in [0,1]$$  \hspace{1cm} (A6.1b)

$$\phi^* = \beta\phi_1 + (1-\beta)\phi_2$$  \hspace{1cm} (A6.1c)

When the function $\phi(\sigma)$ is smooth, it is convex if and only if

$$\phi_2 - \phi_1 \geq (\sigma_2 - \sigma_1)\phi'(\sigma_1)$$  \hspace{1cm} (A6.2)

where $\phi'(\sigma_1)$ is the slope of the function at $\sigma_1$. The idea can be easily generalized for a smooth, multi-variable function.
Appendix 5: Use of Elastic Predictor for Determination of Loading/Unloading Event

The proof may be found in Simo and Hughes (1998), and is summarized here. From the definition of a multi-variable, smooth, convex function, presented in Appendix 4,

$$\phi_{n+1}^r(\sigma_{n+1}^r, \zeta_n) - \phi_{n+1}(\sigma_{n+1}, \zeta_{n+1}) \geq -\Delta \sigma^{pc} \left[ \frac{\partial \phi}{\partial \sigma_{n+1}} - \Delta \zeta \left[ \frac{\partial \phi}{\partial \zeta_{n+1}} \right] \right]$$  \hspace{1cm} (A5.1)

where $\Delta \sigma^{pc}$ is the plastic corrector. Using backward Euler integration method

$$\Delta \sigma = -\int_{t}^{t+\Delta t} \dot{\lambda} \int_{s} = -\Delta \lambda \int_{s} \int_{t} (\text{from Eq. 10.5c})$$

and

$$\Delta \zeta = -\int_{t}^{t+\Delta t} \dot{\lambda} \int_{s} \int_{t} (\text{from Eq. 10.2c})$$

Substituting these into Eq. A5.1, and noting that

$$n_{n+1} = \frac{\partial \phi}{\partial \sigma_{n+1}} \text{ and } K_p = -(\partial \phi)^T s \text{ (Eq. 6.2e)}, \text{ one has,}$$

$$\phi_{n+1}^r(\sigma_{n+1}^r, \zeta_n) - \phi_{n+1}(\sigma_{n+1}, \zeta_{n+1}) \geq \Delta \lambda \left[ n_{n+1}^{T} \int_{s} \int_{t} + K_p^{n+1} \right]$$  \hspace{1cm} (A5.2)

Recalling that $\Delta \lambda \geq 0$, the term on the right hand side is non-negative when the quantity within the bracket is non-negative. For a model employing associated flow rule,

$$n_{n+1} = r_{n+1} \text{. Since the elastic operator is positive definite, the first term is positive because}$$

$$n_{n+1}^{T} \int_{s} \int_{t} + r_{n+1}^{T} \int_{s} \int_{t} \geq 0 \text{. When the model does not permit strain softening, the plastic modulus } K_p \text{. Thus, the quantity within the bracket is definitely non-negative. For other cases (i.e., models using non-associated flow rule and permitting strain softening), the term on the right hand side is still non-negative so far as the terms within the bracket add up to a non-negative number.}$$

Let us assume that the term on the right hand side is non-negative:

$$\phi_{n+1}^r(\sigma_{n+1}^r, \zeta_n) - \phi_{n+1}(\sigma_{n+1}, \zeta_{n+1}) \geq 0$$  \hspace{1cm} (A5.3)

Then a positive value for $\phi_{n+1}^r(\sigma_{n+1}^r, \zeta_n)$ indicates plastic loading. That is,

$$\phi_{n+1}^r(\sigma_{n+1}^r, \zeta_n) > 0 \hspace{1cm} \text{Plastic} \hspace{1cm} (A5.4a)$$

$$\phi_{n+1}^r(\sigma_{n+1}^r, \zeta_n) < 0 \hspace{1cm} \text{Elastic} \hspace{1cm} (A5.4b)$$

Noting that a step is plastic if $\phi_{n+1}(\sigma_{n+1}, \zeta_{n+1}) = 0$, Eqs. A5.4a and A5.4b can be proved as follows:
1. when \( \phi_{n+1}(\sigma_n^r, \zeta_n) < 0 \), it follows from Eq. A5.3 that \( \phi_{n+1}(\sigma_n^r, \zeta_n) < 0 \). Since \( \phi_{n+1}(\sigma_n^r, \zeta_n) = 0 \) for plastic loading, the step is elastic.

2. when \( \phi_{n+1}(\sigma_n^r, \zeta_n) > 0 \), the stress point is outside the current yield surface, and thus must cause plastic deformation, with the consequence that \( \Delta \lambda > 0 \) and hence \( \phi_{n+1}(\sigma_n^r, \zeta_n) = 0 \). The step is, therefore, plastic.

Appendix 6: Simplification of Tensorial Operations due to Symmetries

Let us place the elements of the stress and strain tensors in vectors as

\[
\boldsymbol{\sigma} = \begin{bmatrix} \sigma_{11} & \sigma_{22} & \sigma_{33} & \sigma_{12} & \sigma_{23} & \sigma_{13} \end{bmatrix} \quad (A6.1a)
\]

\[
\boldsymbol{\varepsilon} = \begin{bmatrix} \varepsilon_{11} & \varepsilon_{22} & \varepsilon_{33} & \gamma_{12} & \gamma_{23} & \gamma_{13} \end{bmatrix} \quad (A6.1b)
\]

where \( \gamma_{12} = 2\varepsilon_{12}, \gamma_{23} = 2\varepsilon_{23} \) and \( \gamma_{13} = 2\varepsilon_{13} \). The use of engineering strains renders the stress vector to be different from the strain vector. With this difference in mind, in the following, we will refer to second order tensors as either a “stress-like” quantity or a “strain-like” quantity. We use similar representations for the incremental stresses and strains.

We now present a few tensorial operations that could be simplified into reduced order matrix-vector operations.

**Operation 1**

\[
\beta = A_y B_y
\]

where \( A_y = A_{ij} \) and \( B_y = B_{ij} \)

Then \( A_y \) and \( B_y \) can be placed in vectors like in Eq. A6.1. The required operations can then be performed as

\[
\beta = \mathbf{A} \cdot \mathbf{B}
\]

where \( \mathbf{A} = \{A_{11}, A_{22}, A_{33}, A_{12}, A_{23}, A_{13}\} \)

and \( \mathbf{B} = \{B_{11}, B_{22}, B_{33}, 2B_{12}, 2B_{23}, 2B_{13}\} \)

\( \mathbf{A} \) is a stress-like quantity, and \( \mathbf{B} \) is a strain-like quantity.

**Operation 2**

\[
K_{ij} = L_{ijk} M_{k}\quad (A6.2a)
\]
where \( M_{kk} = M_{kk} \)  

and \( L_{ijk} = L_{jk\ell} \Rightarrow K_{ij} = K_{ji} \)  

Then \( K_{ij} \) and \( M_{ij} \) can be placed in vectors like in Eq. A6.1. The required operations can then be performed as

\[
\begin{bmatrix}
K_{11} & L_{1111} & L_{1122} & L_{1133} & \alpha(L_{1112} + L_{1121}) & \alpha(L_{1112} + L_{1132}) & \alpha(L_{1123} + L_{1132}) \\
K_{22} & L_{2211} & L_{2222} & L_{2233} & \alpha(L_{2212} + L_{2221}) & \alpha(L_{2232} + L_{2233}) & \alpha(L_{2233} + L_{2233}) \\
K_{33} & L_{3311} & L_{3322} & L_{3333} & \alpha(L_{3312} + L_{3321}) & \alpha(L_{3323} + L_{3332}) & \alpha(L_{3333} + L_{3333}) \\
K_{12} & L_{1211} & L_{1222} & L_{1233} & \alpha(L_{1212} + L_{1221}) & \alpha(L_{1223} + L_{1232}) & \alpha(L_{1233} + L_{1233}) \\
K_{23} & L_{2311} & L_{2322} & L_{2333} & \alpha(L_{2312} + L_{2321}) & \alpha(L_{2323} + L_{2332}) & \alpha(L_{2333} + L_{2333}) \\
K_{13} & L_{1311} & L_{1322} & L_{1333} & \alpha(L_{1312} + L_{1321}) & \alpha(L_{1323} + L_{1332}) & \alpha(L_{1333} + L_{1333})
\end{bmatrix}
\begin{bmatrix}
M_{11} \\
M_{22} \\
M_{33} \\
M_{12} \\
M_{23} \\
M_{13}
\end{bmatrix}
\]

(A6.3a)

\[
\mathbf{K} = \mathbf{L} \mathbf{M}
\]

(A6.3b)

When \( M_{ij} \) is a stress-like quantity (Eq. A6.1a), \( \alpha = 1 \), and \( M_{ij} \) is a strain-like quantity (Eq. A6.1b), \( \alpha = 1/2 \). Also when \( L_{ijk} = L_{jk\ell} \), then the terms on the last three columns simplify as \( L_{1112} + L_{1121} = 2L_{1112} \), etc.

Following a similar approach, the operation

\[
K_{k\ell} = L_{ijk} M_{ij}
\]

(A6.3c)

can easily be performed.

**Appendix 7: Gradient of a Yield/Potential Function with Respect to Stresses**

The yield and plastic potentials typically take the same functional form, and thus, the equations derived in this section can be applied to either one.

The yield surface is generally given by the following function

\[
\phi(I, J, \alpha, \zeta) = 0
\]

(A7.1)

By differentiating the yield function with respect to \( \sigma \)

\[
r_{ij} = \frac{\partial \phi}{\partial \sigma_{ij}} = \frac{\partial \phi}{\partial I} \frac{\partial I}{\partial \sigma_{ij}} + \frac{\partial \phi}{\partial J} \frac{\partial J}{\partial \sigma_{ij}} + \frac{\partial \phi}{\partial \alpha} \frac{\partial \alpha}{\partial \sigma_{ij}}
\]

(A7.2)

Three stress invariants \( I, J \) and \( \alpha \) are defined as

\[
I = \sigma_{kl} \delta_{kl}
\]

(A7.3)
\[ s_{ij} = \sigma_{ij} - \frac{1}{3} I \delta_{ij}, \quad J = \left( \frac{1}{2} s_{kl} s_{kl} \right)^{1/2} \]  
(A7.4)

\[ S = \left( \frac{1}{3} s_{ij} s_{jk} s_{ki} \right)^{1/3} \]  
(A7.5)

\[-\frac{\pi}{6} \leq \alpha = \frac{1}{3} \sin^{-1} \left[ \frac{3\sqrt{3}(S)^{3}}{2(J)} \right] \leq \frac{\pi}{6} \]  
(A7.6)

Differentiating these quantities with respect to the stress tensor

\[ \frac{\partial I}{\partial \sigma_{ij}} = \delta_{ij} \]  
(A7.3)

\[ \frac{\partial S_{ij}}{\partial \sigma_{kl}} = \delta_{ik} \delta_{j\ell} - \frac{1}{3} \delta_{ij} \delta_{k\ell} \]  
(A7.4)

From \( 2J^2 = s_{pq} s_{pq} \), one has

\[ 4J \frac{\partial J}{\partial \sigma_{ij}} = (\delta_{pi} \delta_{qj} - \frac{1}{3} \delta_{pq} \delta_{ij}) s_{pq} + ... \]

Noting that \( s_{kk} = 0 \)

\[ 4J \frac{\partial J}{\partial \sigma_{ij}} = 2s_{ij} \]

\[ \frac{\partial J}{\partial \sigma_{ij}} = \frac{s_{ij}}{2J} \]  
(A7.5)

Similarly, noting that \( 3S^3 = s_{pq} s_{qr} s_{rp} \),

\[ 9S^2 \frac{\partial S}{\partial \sigma_{ij}} = (\delta_{pi} \delta_{qj} - \frac{1}{3} \delta_{pq} \delta_{ij}) s_{qr} s_{rp} + ... \]

\[ = (s_{jr} s_{rt} - \frac{1}{3} s_{pr} s_{rp} \delta_{ij}) + ... \]

\[ = 3s_{jr} s_{rt} - \frac{6J^2}{3} \delta_{ij} \]

\[ \frac{\partial S}{\partial \sigma_{ij}} = \frac{1}{3S^2} s_{ir} s_{jr} - \frac{2J^2}{9} \delta_{ij} \]  
(A7.6)

From
\[
\sin 3\alpha = \frac{3\sqrt{3} \, S^3}{2 \, J^3}
\]

one has

\[
3 \cos 3\alpha \frac{\partial \alpha}{\partial \sigma_{ij}} = \frac{3\sqrt{3}}{2} \frac{3S^2}{J^3} \frac{\partial S}{\partial \sigma_{ij}} - \frac{3\sqrt{3}}{2} \frac{3S^3}{J^4} \frac{\partial J}{\partial \sigma_{ij}}
\]

\[
= \frac{3\sqrt{3}}{2} \frac{3S^2}{J^3} \left[ \frac{1}{3S^2} s_{\nu\tau} s_{\tau\nu} - \frac{2J^2}{9} \delta_{ij} \right] - \frac{3\sqrt{3}}{2} \frac{3S^3}{J^4} \frac{s_{ij}}{2J} \quad \text{(using Eqs. A7.5 and A7.6)}
\]

\[
\frac{\partial \alpha}{\partial \sigma_{ij}} = \frac{\sqrt{3}}{2J \cos 3\alpha} \left[ \frac{s_{\nu\tau} s_{\tau\nu}}{J^2} - \frac{2}{3} \delta_{ij} - \frac{3}{2} \left( \frac{S}{J} \right)^3 \frac{s_{ij}}{J} \right]
\]

(A7.7)

Synthesizing the above equations

\[
r_{ij} = \frac{\partial \phi}{\partial I} \delta_{ij} + \frac{\partial \phi}{\partial J} \frac{\sqrt{3}}{2J} + \frac{\partial \phi}{\partial \alpha} \frac{\sqrt{3}}{2J \cos 3\alpha} \left[ \frac{s_{\nu\tau} s_{\tau\nu}}{J^2} - \frac{2}{3} \delta_{ij} - \frac{3}{2} \left( \frac{S}{J} \right)^3 \frac{s_{ij}}{J} \right]
\]

(A7.8)

Eq. A7.8 is quite general and can be used with any surface that is represented in a functional form given by Eq. A7.1. Depending on the specific shape of the surface, \( \partial_i \phi \), \( \partial_J \phi \), \( \partial \alpha \phi \) and \( \partial_a \phi \) will vary.

**Appendix 8: Equations Related to Cam-Clay Model**

**Appendix 8.1: Surface Properties for Cam-Clay Ellipse: First Gradients of Yield Function with Respect to Invariants**

The Cam-clay ellipse is given in a 3-invariant space as

\[
\phi(I,J,\alpha, p_0) = I^2 + \left( \frac{J}{N} \right)^2 - 3p_0J = 0
\]

(A8.1)

where \( N = \frac{M}{3\sqrt{3}} \); \( M = g(n,\alpha)M_c \); \( g(n,\alpha) = \frac{2n}{1 + n - (1 - n)\sin 3\alpha} \);

and \( n = \frac{M_c}{M_c} \)

(A8.2)

Differentiating Eq. A8.1

\[
\frac{\partial \phi}{\partial I} = 2I - 3p_0
\]

(A8.3)

\[
\frac{\partial \phi}{\partial J} = \frac{2J}{N^2}
\]

(A8.4)
\[
\frac{\partial \phi}{\partial \alpha} = -\frac{2J^2}{N^3} \frac{\partial N}{\partial \alpha} = -\frac{J}{N} \frac{\partial \phi}{\partial N} \frac{\partial N}{\partial J} \frac{\partial \alpha}{\partial \alpha} \quad \text{(using Eq. A8.4)}
\]

From Eq. A8.2
\[
\frac{\partial N}{\partial \alpha} = \frac{M_c}{3\sqrt{3}} \frac{\partial g}{\partial \alpha} = \frac{M_c}{3\sqrt{3}} \left[ \frac{(2n)(1-n)(3\cos 3\alpha)}{\left[1 + n - (1-n)\sin 3\alpha\right]^2} \right] = \frac{3N(1-n)\cos 3\alpha}{1 + n - (1-n)\sin 3\alpha} \quad \text{(A8.5)}
\]

Thus,
\[
\frac{\partial \phi}{\partial \alpha} = Jg \frac{\partial \phi}{\partial J} \quad \text{where} \quad g = -\frac{3(1-n)\cos 3\alpha}{1 + n - (1-n)\sin 3\alpha} \quad \text{(A8.6)}
\]

Also,
\[
\frac{\partial \phi}{\partial I_0} = -I \quad \text{(A8.7)}
\]

Eqs. A8.3, A8.4, A8.6 and A8.7 furnish the gradients \( \partial \phi \), \( \partial J \phi \), \( \partial \alpha \phi \), and \( \partial_t \phi \) respectively.

**Appendix 8.2. Surface Properties for Cam-Clay Ellipse:**

**Second Gradients of Yield Function with Respect to Stresses**

Repeating Eq. A8.8
\[
\frac{\partial \phi}{\partial I} = 2I - 3p_0 \quad \text{(A8.9)}
\]

\[
\frac{\partial \phi}{\partial J} = \frac{2J}{N^2} \quad \text{(A8.10)}
\]

\[
\frac{\partial \phi}{\partial \alpha} = Jg \frac{\partial \phi}{\partial J} \quad \text{where} \quad g = -\frac{3(1-n)\cos 3\alpha}{1 + n - (1-n)\sin 3\alpha} \quad \text{(A8.11)}
\]

\[
\frac{\partial \phi}{\partial I_0} = -I \quad \text{(A8.12)}
\]

The 2\textsuperscript{nd} term in Eq. A8.8 is to be excluded for \( J = 0 \), and the 3\textsuperscript{rd} term for \( J = 0 \) and/or \( \alpha = \pi/6 \) or \( -\pi/6 \).
From Eqs. A7.3-A7.7

\[
\frac{\partial I}{\partial \sigma_{ij}} = \delta_{ij} \quad (A8.13)
\]

\[
\frac{\partial S_{ij}}{\partial \sigma_{kl}} = \delta_{ik} \delta_{jl} - \frac{1}{3} \delta_{ij} \delta_{kl} \quad (A8.14)
\]

\[
\frac{\partial J}{\partial \sigma_{ij}} = \frac{s_{ij}}{2J} \quad (A8.15)
\]

\[
\frac{\partial S}{\partial \sigma_{ij}} = \frac{1}{3S^2} s_{ij} s_{jr} - \frac{2J^2}{9} \delta_{ij} \quad (A8.16)
\]

\[
\frac{\partial \alpha}{\partial \sigma_{ij}} = \frac{\sqrt{3}}{2J \cos 3\alpha} \left[ \frac{s_{ij} s_{jr}}{J^2} - \frac{2}{3} \delta_{ij} - \frac{3}{2} \left( \frac{S}{J} \right)^3 \delta_{ij} \right] \quad (A8.17)
\]

With the aid of equations in Appendix 8.1,

\[
N = \frac{M}{3\sqrt{3}} = \frac{g(n,\alpha)M_c}{3\sqrt{3}}, \text{ where } g(n,\alpha) = \frac{2n}{1 + n -(1-n)\sin 3\alpha} \quad (A8.18)
\]

\[
\frac{\partial g}{\partial \alpha} = \frac{(2n)(1-n)(3\cos 3\alpha)}{[1 + n -(1-n)\sin 3\alpha]^2} = \frac{3(1-n)g^2 \cos 3\alpha}{2n} \quad (A8.19)
\]

\[
\frac{\partial N}{\partial \sigma_{pq}} = \frac{M_c}{3\sqrt{3}} \frac{\partial g(n,\alpha)}{\partial \sigma_{pq}} = \frac{M_c}{3\sqrt{3}} \frac{\partial g(n,\alpha)}{\partial \alpha} \frac{\partial \alpha}{\partial \sigma_{pq}}
\]

\[
= \frac{M_c}{3\sqrt{3}} \frac{3(1-n)g^2 \cos 3\alpha}{2n} \frac{\partial \alpha}{\partial \sigma_{pq}}
\]

\[
= \frac{M_c}{2\sqrt{3}} \frac{1-n}{n} g^2 \cos 3\alpha \frac{\partial \alpha}{\partial \sigma_{pq}}
\]

\[
= \frac{3N}{2} \frac{1-n}{n} g \cos 3\alpha \frac{\partial \alpha}{\partial \sigma_{pq}} \quad (A8.20)
\]

From Eq. A8.11

\[
\bar{g} = -\frac{3(1-n) \cos 3\alpha}{1 + n -(1-n)\sin 3\alpha} = -\frac{3(1-n)}{2n} g \cos 3\alpha
\]
\[
\frac{\partial \sigma_{pq}}{\partial \sigma_{pq}} = \frac{9(1-n)}{2n} \sin 3\alpha \frac{\partial \alpha}{\partial \sigma_{pq}} - \frac{3(1-n)}{2n} \cos 3\alpha \frac{\partial g}{\partial \sigma_{pq}} \\
= \frac{9(1-n)}{2n} \sin 3\alpha \frac{\partial \alpha}{\partial \sigma_{pq}} - \frac{3(1-n)}{2n} \cos 3\alpha \frac{\partial g}{\partial \alpha} \frac{\partial \alpha}{\partial \sigma_{pq}} \\
= \frac{9(1-n)}{2n} \sin 3\alpha \frac{\partial \alpha}{\partial \sigma_{pq}} - \frac{3(1-n)}{2n} \cos 3\alpha \left\{ \frac{3(1-n)g^2 \cos 3\alpha}{2n} \right\} \frac{\partial \alpha}{\partial \sigma_{pq}} \\
\text{(using Eq. A8.19)} \\
\]

By differentiation of Eqs. A8.9 to A8.12

\[
\frac{\partial [\partial_j \phi]}{\partial \sigma_{pq}} = 2\delta_{pq} \quad \text{(A8.22)} \\
\frac{\partial [\partial_j \phi]}{\partial \sigma_{pq}} = \frac{2}{N^2} \frac{\partial J}{\partial \sigma_{pq}} - \frac{4J}{N^2} \frac{\partial N}{\partial \sigma_{pq}} \\
= \frac{2}{N^2} \frac{S_{pq}}{2J} - \frac{4J}{N^2} \frac{3N}{2} \frac{1-n}{n} g \cos 3\alpha \frac{\partial \alpha}{\partial \sigma_{pq}} \quad \text{(using Eqs. A8.15 and A8.20)} \\
= \frac{1}{N^2} \frac{S_{pq}}{J} - \frac{6J}{N^2} \frac{1-n}{n} g \cos 3\alpha \frac{\partial \alpha}{\partial \sigma_{pq}} \quad \text{(A8.23)} \\
\frac{\partial [\partial_j \phi]}{\partial \sigma_{pq}} = Jg \frac{\partial [\partial_j \phi]}{\partial \sigma_{pq}} + g [\partial_j \phi] \frac{\partial J}{\partial \sigma_{pq}} + J [\partial_j \phi] \frac{\partial g}{\partial \sigma_{pq}} \\
= Jg \frac{\partial [\partial_j \phi]}{\partial \sigma_{pq}} + g [\partial_j \phi] \frac{S_{pq}}{2J} + J [\partial_j \phi] \left\{ \frac{9(1-n)}{2n} \sin 3\alpha - \frac{9(1-n)^2}{4n^2} g^2 \cos 3\alpha \right\} \frac{\partial \alpha}{\partial \sigma_{pq}} \\
\text{(using Eqs. A8.15 and A8.21)} \quad \text{(A8.24)}
\]
Now, let us differentiate Eq. 8.8. Let us denote the terms associated with \( \partial_{\gamma} \phi \), \( \partial_{\gamma} \phi \) and \( \partial_{\alpha} \phi \) separately by \( L_{ijpq}^1 \), \( L_{ijpq}^2 \) and \( L_{ijpq}^3 \) respectively. Writing \( r_{ij} \) as
\[
 r_{ij} = (\partial_{\gamma} \phi) a_{ij} + (\partial_{\gamma} \phi) b_{ij} + (\partial_{\alpha} \phi) c_{ij}
\]
where
\[
 a_{ij} = \frac{\partial I}{\partial \sigma_{ij}} = \delta_{ij} \\
 b_{ij} = \frac{\partial J}{\partial \sigma_{ij}} = \frac{s_{ij}}{2J} \\
 c_{ij} = \frac{\partial \alpha}{\partial \sigma_{ij}} = \frac{\sqrt{3}}{2J \cos3\alpha} \left[ s_{ij} s_{ij} - \frac{2}{3} \delta_{ij} - \frac{3(S)^3}{2J} \right]
\]
it follows:
\[
 \begin{align*}
 \frac{\partial r_{ij}}{\partial \sigma_{pq}} &= L_{ijpq}^1 + L_{ijpq}^2 + L_{ijpq}^3 \\
 L_{ijpq}^1 &= \frac{\partial (\partial_{\gamma} \phi)}{\partial \sigma_{pq}} a_{ij} + (\partial_{\gamma} \phi) L_{ijpq}^0 \\
 L_{ijpq}^0 &= \frac{\partial a_{ij}}{\partial \sigma_{pq}} \\
 L_{ijpq}^2 &= \frac{\partial (\partial_{\gamma} \phi)}{\partial \sigma_{pq}} b_{ij} + (\partial_{\gamma} \phi) L_{ijpq}^0 \\
 L_{ijpq}^0 &= \frac{\partial b_{ij}}{\partial \sigma_{pq}} \\
 L_{ijpq}^3 &= \frac{\partial (\partial_{\alpha} \phi)}{\partial \sigma_{pq}} c_{ij} + (\partial_{\alpha} \phi) L_{ijpq}^0 \\
 L_{ijpq}^0 &= \frac{\partial c_{ij}}{\partial \sigma_{pq}}
\end{align*}
\]
It is easy to see that
\[
 L_{ijpq}^0 = 0
\]  
(A8.25b)
\[
 L_{ijpq}^2 = \frac{1}{2J} \frac{\partial s_{ij}}{\partial \sigma_{pq}} - \frac{1}{2J^2} \frac{\partial J}{\partial \sigma_{pq}} s_{ij}
\]
\[
 = \frac{1}{2J} \left[ \delta_{ij} \delta_{pq} - \frac{1}{3} \delta_{ij} \delta_{pq} \right] - \frac{1}{2J^2} \frac{s_{pj}}{2J} s_{ij}
\]
(using Eqs. A8.14 and A8.15)
\[
 (A8.26)
\]
Writing \( c_{ij} \) as
\[
 c_{ij} = \frac{\sqrt{3}}{2 \cos3\alpha} A_{ij} = a_{0} A_{ij}
\]
where \( A_{ij} = \frac{s_{ij} s_{ij}}{J^3} - \frac{2}{3} \delta_{ij} \frac{1}{J} - \frac{3}{2J^3} S_{ij} \), and \( a_{0} = \frac{\sqrt{3}}{2 \cos3\alpha} \) then
\[
 L_{ijpq}^3 = \frac{\partial c_{ij}}{\partial \sigma_{pq}} = \frac{3\sqrt{3} \sin3\alpha}{2 \cos^23\alpha} \frac{\partial \alpha}{\partial \sigma_{pq}} A_{ij}
\]
\[ + \frac{a_0}{J^3} \frac{\partial s_{ir}}{\partial \sigma_{pq}} s_{rq} + \frac{a_0}{J^3} \frac{\partial s_{rij}}{\partial \sigma_{pq}} s_{iur} + a_0 \left[ \frac{15}{2} \frac{S^3}{J^6} s_{ij} + \frac{2}{3} \frac{1}{J^2} \delta_{ij} - \frac{3}{J^5} \frac{S_{ij}}{s_{ij}} s_{ij} \right] \frac{\partial J}{\partial \sigma_{pq}} \]
\[ - \frac{9a_0}{2} \frac{S^2}{J^3} s_{ij} \frac{\partial S}{\partial \sigma_{pq}} - \frac{3}{2} \frac{S^3}{J^5} \frac{\partial s_{ij}}{\partial \sigma_{pq}} \]  
(A8.27)

With the aid of Eqs. A8.14, A8.15 and A8.16, and noting
\[ \frac{\partial s_{ir}}{\partial \sigma_{pq}} s_{rq} = s_{pq} \delta_{ir} - \frac{1}{3} s_{ij} \delta_{pq} ; \quad \frac{\partial s_{rij}}{\partial \sigma_{pq}} s_{iur} = s_{pq} \delta_{rij} - \frac{1}{3} s_{ij} \delta_{pq} ; \quad \frac{\sqrt{3}}{2 \cos 3\alpha} A_y = \frac{\partial \alpha}{\partial \sigma_{pq}} \]  
(A8.28)

\( L_{ijpq}^{30} \) can be evaluated from Eq. A8.27.

Using the expressions for \( \frac{\partial [\partial, \phi]}{\partial \sigma_{pq}} \), \( \frac{\partial [\partial, \phi]}{\partial \sigma_{pq}} \) and \( \frac{\partial [\partial, \phi]}{\partial \sigma_{pq}} \) from Eqs. A8.22, A8.23 and A8.24, \( L_{ijpq}^{1} \), \( L_{ijpq}^{2} \) and \( L_{ijpq}^{3} \) can now be evaluated. \( \frac{\partial r_{ij}}{\partial \sigma_{pq}} \) is then calculated from A8.25a.

Also note that from Eq. A8.8 that
\[ r_{kk} = \frac{3}{2} \frac{\partial \phi}{\partial J} \]  
(A8.29a)
\[ \frac{\partial r_{kk}}{\partial \sigma_{pq}} = 3 \frac{\partial [\partial, \phi]}{\partial \sigma_{pq}} = 6 \delta_{pq} \]  
(A8.29b)
\[ \frac{\partial r_{ij}}{\partial p_0} = \frac{\partial [\partial, \phi]}{\partial p_0} \delta_{ij} + \frac{\partial [\partial, \phi]}{\partial p_0} s_{ij} \frac{\sqrt{3}}{2J} + \frac{\partial [\partial, \phi]}{\partial p_0} \frac{\sqrt{3}}{2J \cos 3\alpha} \left[ \frac{s_{ij} s_{ij}}{J^2} - \frac{2}{3} \delta_{ij} - \frac{3}{2J} \right] \]  
\[ = -3 \delta_{ij} \]  
(A8.29c)

Defining the deviatoric part of \( r_{ij} \) as
\[ d_{ij} = r_{ij} - \frac{1}{3} r_{kk} \delta_{ij} \]  
(A8.29d)
then
\[ \frac{\partial d_{ij}}{\partial \sigma_{pq}} = \frac{\partial r_{ij}}{\partial \sigma_{pq}} - 2 \delta_{pq} \delta_{ij} \]  
(A8.29e)
\[ \frac{\partial d_{ij}}{\partial p_0} = -3 \delta_{ij} + 3 \delta_{ij} = 0 \]  
(A8.29f)
Appendix 8.3. Properties Associated with Cam-Clay Hardening Relations

\[
\dot{\lambda} = \dot{p}_0 = C_1 p_o \dot{\varepsilon}_v = C_1 p_o r_{kk} \dot{\lambda} = s \dot{\lambda}
\]  
(A8.30a)

where \( C_1 = \frac{1 + e_o}{\lambda^* - \kappa} \) is a constant, and \( s = C_1 p_o r_{kk} \)  
(A8.30b)

Gradients of \( s \) are

\[
\frac{\partial s}{\partial \sigma_{pq}} = C_1 p_o \frac{\partial r_{kk}}{\partial \sigma_{pq}} = 3 C_1 p_o \frac{\partial [\partial_{ij} \phi]}{\partial \sigma_{pq}} \text{ (using Eq. A8.29b)}
\]  
(A8.31)

\[
\frac{\partial s}{\partial \zeta} = \frac{\partial s}{\partial p_0} = C_1 r_{kk} + C_1 p_o \frac{\partial r_{kk}}{\partial p_0} = 3 C_1 [\partial_{ij} \phi] - 9 C_1 p_0 \text{ (Using Eqs. A8.29a and A8.9)}
\]

Appendix 8.4. Exact Elastic Predictor for Cam-Clay Elasticity

Splitting elastic constitutive relations into volumetric and deviatoric parts

\[
\dot{p} = K \dot{\varepsilon}_{kk}
\]  
(A8.32a)

and \( \dot{s}_{ij} = 2 G \dot{e}_{ij} \)  
(A8.32b)

where \( \dot{s}_{ij} = \dot{\sigma}_{ij} - \dot{p} \delta_{ij} \)  
(A8.33a)

and \( \dot{e}_{ij} = \dot{\varepsilon}_{ij} - \frac{1}{3} \dot{\varepsilon}_{kk} \delta_{ij} \)  
(A8.33b)

Using the relationships between \( K \) and \( G \) , and \( p \):

\[
K = \overline{K}_0 p \quad \text{and} \quad G = \overline{G}_0 p
\]  
(A8.34a)

where \( \overline{K}_0 = \frac{1 + e_o}{\kappa} \) and \( \overline{G}_0 = \frac{3(1 - 2 \nu)}{2(1 + \nu)} \overline{K}_0 \)  
(A8.34b)

Let \( \delta \varepsilon_{ij} = x \Delta \varepsilon_{ij}, \ x \in [0,1] \)  
(A8.35)

with \( x = 0 \) at the beginning of the increment, \( x = 1 \) at the end of the increment and \( \Delta \varepsilon_{ij} \) is the total strain increment, which is a constant.

Combining Eqs. A8.32a and A8.34a and integrating the rate equations

\[
\int_{\dot{p}^p}^{p} \dot{p} = \overline{K}_0 \int_{\dot{\varepsilon}_{kk}}^{\varepsilon_{kk}} \dot{\varepsilon}_{kk}
\]  
(A8.36a)

\[
p = p^n \exp[\overline{K}_0 x \Delta \varepsilon_{kk}] \quad \text{or} \quad p = p^n \exp[\overline{K}_0 x \Delta \varepsilon_{kk}]
\]  
(A8.36b)
Noting that $\dot{\varepsilon}_{ij} = x \Delta \varepsilon_{ij}$, and
\begin{equation}
\dot{\varepsilon}_{ij} = x \Delta \varepsilon_{ij} \tag{A8.37a}
\end{equation}
Eq. A.8.32b is written as
\begin{equation}
\dot{s}_{ij} = 2G_0 \dot{p} \Delta \varepsilon_{ij} \dot{x} = 2G_0 p^n e^{\eta \Delta \varepsilon_{kk}} \Delta \varepsilon_{ij} \dot{x} \tag{A8.37b}
\end{equation}

Integrating this equation,
\begin{equation}
s_{ij} = s_{ij}^n + \frac{2G_0 p^n}{K_0} e^{\eta \Delta \varepsilon_{kk}} - 1 \Delta \varepsilon_{ij} \tag{A8.38}
\end{equation}
The stresses at the end of the increment are obtained from Eq. A8.36b and A8.38 for $x = 1$ as
\begin{equation}
p_{ij}^{n+1} = p^n \exp\left[K_0 \Delta \varepsilon_{kk}\right] \tag{A8.39a}
\end{equation}
\begin{equation}
s_{ij}^{n+1} = s_{ij}^n + \frac{2G_0 p^n}{K_0} e^{\eta \Delta \varepsilon_{kk}} - 1 \Delta \varepsilon_{ij} \tag{A8.39b}
\end{equation}
Note that when $\Delta \varepsilon_{kk} = 0$, Eq. A8.39b is not valid since the factor (2 times the average shear modulus $G_{ave}$) $2G_{ave} = \frac{2G_0 p^n}{K_0} e^{\eta \Delta \varepsilon_{kk}} - 1$ goes to $\infty$. Applying L’Hospital’s rule, it is seen that
\begin{equation}
2G_{ave} \Rightarrow 2G_0 p^n = 2G_n \text{ as } \Delta \varepsilon_{kk} \Rightarrow 0 \tag{A8.40}
\end{equation}
This must be recognized and dealt with in computer implementations. For example, it could be handled as follows:
\begin{equation}
G_{ave} = \frac{G_0 p^n}{K_0} e^{\eta \Delta \varepsilon_{kk}} - 1 \text{ for } \Delta \varepsilon_{kk} \geq TOL \tag{A8.41a}
\end{equation}
and
\begin{equation}
G_{ave} = \frac{G_0 p^n}{K_0(TOL)} - 1 \text{ for } \Delta \varepsilon_{kk} < TOL \tag{A8.41b}
\end{equation}
where $TOL$ is a suitable small number

The stress tensor at the end of the increment is obtained as
\begin{equation}
\sigma_{ij}^{n+1} = s_{ij}^{n+1} + p^{n+1} \delta_{ij} \tag{A8.42}
\end{equation}

**Appendix 9: Parameters for Plane Strain Analysis**

In this section, we derive expressions for the model parameters that control the strength of materials that is suitable for use in two-dimensional plane strain analyses. Drucker and
Prager (1952) had done this for the Drucker-Prager model; we extend their work to von Mises (Chapter 11) and Cam-clay (Chapter 12) models as well.

In plane strain,
\[ \varepsilon_{13} = \varepsilon_{23} = \varepsilon_{33} = 0. \quad (A9.1) \]

It is shown in Chapter 9 that when the material experiences flow, the rate of change of stress is zero; i.e., \( \sigma = 0 \) (See Section 9.3.7., Eq. 9.52). This leads to zero strain rates for the elastic strains, and hence
\[ \dot{\varepsilon}_{ij} = \dot{\varepsilon}_{ij}^p \quad (A9.2) \]

where \( \varepsilon_{ij}^p \) is the plastic strain. Hence Eq. A9.1 becomes
\[ \dot{\varepsilon}_{13} = \dot{\varepsilon}_{23} = \dot{\varepsilon}_{33} = 0 \quad (A9.3) \]

The plastic strain rate is in general expressed as (Eq. 9.19a)
\[ \dot{\varepsilon}_{ij}^p = \lambda r_i \quad (A9.4) \]

where \( \lambda \) is the consistency parameter (or loading index) and \( r_i \) is the direction of plastic strain rate, which depends on the specific constitutive model employed. Combining Eqs. A9.3 and A9.4
\[ r_{13} = r_{23} = r_{33} = 0 \quad (A9.5) \]

(a) Potential Failure Mechanism           (b) Element at C       (c) Mohr Circle for Stresses at C

Fig. A9.1. Potential Failure Mechanism for the Metal Forging/Bearing Capacity Problem

Referring to Fig. 9.1a, for the metal forging problem (Chapter 11) or the bearing capacity problem (Chapters 12 and 13), let us assume that the failure takes place by plastic flow at points along a failure surface such as the one shown. The Mohr circle for the stresses acting on an element on the failure surface (Point C, Fig. A9.1b) is shown in Fig. A9.1c. Denoting the distance to the center as \( x_0 \) and the radius as \( r_0 \), it is seen that
\[ x_0 = \frac{1}{2}(\sigma_{11} + \sigma_{22}) \quad (A9.6a) \]
\[ r_0 = \left[ \frac{(\sigma_{11} - \sigma_{22})^2}{2} + \sigma_{12}^2 \right]^{1/2} \quad (A9.6b) \]
9.1. The von Mises Model (Chapter 11)

For the von Mises model (Eq. 11.8b):
\[ r_y = 2s_y \]  

(A9.7)

Hence from Eq. A9.5
\[ s_{13} = s_{23} = s_{33} = 0 \]  

(A9.8)

Since \( s_{ij} = \sigma_{ij} \) for \( i \neq j \), one has \( \sigma_{13} = \sigma_{23} = 0 \). For the third condition in Eq. A9.7:
\[ s_{33} = \sigma_{33} - \frac{I}{3} = \sigma_{33} - \frac{1}{3}(\sigma_{11} + \sigma_{22} + \sigma_{33}) = 0 \]
\[ \sigma_{33} = \frac{1}{2}(\sigma_{11} + \sigma_{22}) \]
\[ I = 3\sigma_{33} = \frac{3}{2}(\sigma_{11} + \sigma_{22}) \]  

(A9.9a)

Similarly, noting Eq. A9.8,
\[ 2J^2 = s_{xy}s_{xy} = s_{11}^2 + s_{22}^2 + 2s_{12}^2 \]
\[ = \left( \sigma_{11} - \frac{1}{2}(\sigma_{11} + \sigma_{22}) \right)^2 + \left( \sigma_{11} - \frac{1}{2}(\sigma_{11} + \sigma_{22}) \right)^2 + 2\sigma_{12}^2 \]
\[ = 2\left[ \left( \frac{\sigma_{11} - \sigma_{22}}{2} \right)^2 + \sigma_{12}^2 \right] \]  

(A9.9b)

\[ J = \left[ \left( \frac{\sigma_{11} - \sigma_{22}}{2} \right)^2 + \sigma_{12}^2 \right]^{1/2} \]

With the aid of Eqs. A9.6a and A9.6b, Eqs. A9.9a and A9.9b become (Fig. A9.1bc):
\[ I = 3x_0 \]  

(A9.10a)
\[ J = r_0 \]  

(A9.10b)

Denoting the uniaxial failure stress as \( \sigma_f \) (which is also equal to twice the failure shear stress \( 2\tau_f \), Eq. 9.59b), one has
\[ q = \sigma_1 - \sigma_2 = \sigma_f \]
where \( \sigma_1 \) and \( \sigma_2 \) are the principal stresses. Referring to Fig. A9.1c
\[ \sigma_1 - \sigma_2 = 2r_0 \]
Hence
\[ r_0 = \frac{\sigma_f}{2} \]  

(A9.11)

From the equation of the failure surface (Eq. 11.3)
\[ s_{xy}s_{xy} = \frac{2}{3}k^2 \Rightarrow J = \frac{k}{\sqrt{3}} \]  

(A9.12)

Combining Eqs. A9.10a, A9.11 and A9.12:
\[ k = \frac{\sqrt{3}}{2} \sigma_f = 0.866\sigma_f \quad (A9.13) \]

Noting that for 3D analyses, Eq. 9.58b implies that \( k = \sigma_f \), we can summarize the results as

Plane strain: \[ k = k_{PS} = \frac{\sqrt{3}}{2} \sigma_f = 0.866\sigma_f \quad (A9.14a) \]

Three dimensions: \[ k = k_{3D} = \sigma_f \quad (A9.14b) \]

Remark: Geotechnical Engineering Applications

In geotechnical engineering applications, the von Mises model is sometimes used to analyze a cohesion soil with the assumption of zero friction. In this case, the radius of the Mohr circle (Fig. A9.1c) is the cohesion \( c \). Equations A9.14a and A9.14b then become

Plane strain: \[ k = k_{PS} = c\sqrt{3} = 1.732c \quad (A9.15a) \]

Three dimensions: \[ k = k_{3D} = 2c \quad (A9.15b) \]

9.2. The Modified Cam-Clay Model (Chapter 12)

We will only consider the “ultimate” failure, which always occur on the critical state line. The failure points lie on the critical state line, where during flow, \( r_{kk} = 0 \). By the Mohr-Coulomb failure criterion, the shear strength can be represented by

\[ \tau_f = \sigma \tan \phi \quad (A9.16) \]

For the modified Cam-clay model, assuming a circular failure surface in the octahedral plane (Eq. 12.10):

\[ r_y = \frac{\partial \phi}{\partial I} \delta_y + \frac{\partial \phi}{\partial J} \frac{s_{yy}}{2J} \]

When flow occurs, \( r_{kk} = 0 \) and hence (making use of A8.10)

\[ r_y = \frac{\partial \phi}{\partial J} \frac{s_{yy}}{2J} = \frac{s_{yy}}{N^2} \]

For plane strain, as in the case with the von Mises model, Eq. A9.5 becomes

\[ s_{13} = s_{23} = s_{33} = 0 \]

which leads to identical expressions for \( I \) and \( J \) as in the preceding section (Eqs. A9.10a and A9.10b):

\[ I = 3x_0 \quad (A9.16a) \]

\[ J = r_0 \quad (A9.16b) \]

Now we use the Mohr-Coulomb criterion to find the plane strain parameters. During the footing failure, slip occurs along the failure surface and hence the plane parallel to the failure surface (the plane 2) becomes the failure plane. The failure envelop must pass
through Point A in Fig. A9.1. Since cohesion is zero for the case considered here, the failure envelop also must through the origin in Fig. A9.1c. Hence it is easy to see

\[ r_0 = x_0 \sin \phi \quad \text{(A9.17)} \]

Combining Eqs. A9.16a, A9.16b and A9.17, one has

\[ N = \left( \frac{J}{I} \right)_{\text{CSL}} = \frac{r_0}{3x_0} = \frac{\sin \phi}{3} \]

and \[ M = 3\sqrt{3}N = \sqrt{3} \sin \phi \]

Summarizing the results:

Plane strain: \[ M_{PS} = \sqrt{3} \sin \phi \quad \text{(A9.18a)} \]

Three dimensions (Eq. 9.80): \[ M_{e,3D} = \frac{6 \sin \phi}{3 - \sin \phi}; \quad M_{e,3D} = \frac{6 \sin \phi}{3 + \sin \phi} \quad \text{(A9.18b)} \]

where \( M_{e,3D} \) and \( M_{e,3D} \) values of \( M \) to be used in triaxial compression and extension respectively in 3D analyses (according to the Mohr-Coulomb criterion).

For example, for \( \phi = 30^\circ \), \( M_{PS} = 0.866 \), \( M_{e,3D} = 1.2 \) and \( M_{e,3D} = 0.857 = 0.714M_{e,3D} \). It is clear that the plane strain and 3D parameters are very different from each other.

9.3. The Drucker-Prager Model (Chapter 13)

The Drucker-Prager model may be used for a general \( c - \phi \) material. By the Mohr-Coulomb failure criterion, the shear strength can be represented by

\[ \tau_f = c + \sigma \tan \phi \quad \text{(A9.19)} \]

For the Drucker-Prager model model, assuming a circular failure surface in the octahedral plane (Eq. 12.10):

\[ r_y = \frac{\partial \phi}{\partial I} \delta_y + \frac{\partial \phi}{\partial J} \frac{s_y}{2J} \]

Noting that the hardening parameter \( m = 1 \) at failure

\[ r_y = -d_0 \alpha \delta_y + \frac{s_y}{2J} \]

For plane strain, Eq. A9.5 becomes

\[ s_{13} = s_{23} = 0 \]
\[ s_{33} = 2d_0 \alpha J \quad \text{(A9.20)} \]

Using \( I = \sigma_{11} + \sigma_{22} + \sigma_{33} \Rightarrow \sigma_{33} = -\left( \sigma_{11} + \sigma_{22} \right) \)

Eq. A9.20 is expressed as

\[ I - \sigma_{11} + \sigma_{22} - \frac{I}{3} = 2d_0 \alpha J \]
\[ I = 3 \left[ \frac{1}{2} (\sigma_{11} + \sigma_{22}) + d_0 \alpha J \right] = 3x_0 + 3d_0 \alpha J \]  \hspace{1cm} (A9.21)

where \( x_0 \) is defined in Eq. A9.6a. Similarly, \( J \) is expressed as

\[ 2J^2 = s_{11}^2 + s_{22}^2 + s_{33}^2 + 2s_{12}^2 \]

\[ 2J^2 = \left[ \sigma_{11} - \frac{\sigma_{22} + \sigma_{33}}{2} - d_0 \alpha J \right]^2 + \left[ \sigma_{11} - \frac{\sigma_{22} + \sigma_{33}}{2} - d_0 \alpha J \right]^2 + 4d_0^2 \alpha^2 J^2 + 2\sigma_{12}^2 \]

\[ J^2 = \left[ \left( \frac{\sigma_{11} - \sigma_{22}}{2} \right)^2 + \sigma_{12}^2 \right] + 3d_0^2 \alpha^2 J^2 \]

\[ J = \frac{r_0}{(1 - 3d_0^2 \alpha^2)^{1/2}} \]  \hspace{1cm} (A9.22)

where \( r_0 \) is defined by Eq. A9.6b. At failure, the stress point must lie on the Drucker-Prager failure surface, which is given by

\[ J = \alpha I + k \]

Substituting for \( I \) from Eqs. A9.21,

\[ J = \alpha (3x_0 + 3d_0 \alpha J) + k \]

\[ J(1 - 3d_0^2 \alpha^2) = 3\alpha x_0 + k \]

Substituting for \( J \) from Eqs. A9.22,

\[ \frac{r_0}{(1 - 3d_0^2 \alpha^2)^{1/2}} (1 - 3d_0^2 \alpha^2) = 3\alpha x_0 + k \]

\[ r_0 = \frac{3\alpha (1 - 3d_0^2 \alpha^2)^{1/2}}{(1 - 3d_0^2 \alpha^2)} x_0 + k \frac{(1 - 3d_0^2 \alpha^2)^{1/2}}{(1 - 3d_0^2 \alpha^2)} \]  \hspace{1cm} (A9.23)

According to the Mohr-Coulomb criterion, it is easy to show that for a \( c - \phi \) material:

\[ r_0 = x_0 \sin \phi + c \cos \phi \]  \hspace{1cm} (A9.24)

Comparing Eqs. A9.23 and A9.24:

\[ \frac{3\alpha (1 - 3d_0^2 \alpha^2)^{1/2}}{(1 - 3d_0^2 \alpha^2)} = \sin \phi \]  \hspace{1cm} (A9.25a)

\[ \frac{k(1 - 3d_0^2 \alpha^2)^{1/2}}{(1 - 3d_0^2 \alpha^2)} = c \cos \phi \]  \hspace{1cm} (A9.25b)

**Case A: Associated Flow Rule: \( d_0 = 1 \)**

For this case, Eq. A9.25a becomes

\[ \frac{3\alpha}{(1 - 3d_0^2 \alpha^2)^{1/2}} = \sin \phi \]
\[9\alpha^2 = (1 - 3d_0\alpha^2) \sin^2 \phi\]
\[\alpha = \frac{\sin \phi}{\sqrt{3(3 + \sin^2 \phi)^{1/2}}}\]

From the relation \(\sin^2 \phi = \frac{\tan^2 \phi}{1 + \tan^2 \phi}\), it is easy to show that the above equation becomes
\[\alpha = \frac{\tan \phi}{(9 + 12 \tan^2 \phi)^{1/2}}\]

(A9.26a)

From Eq. A9.25b:
\[\frac{k}{(1 - 3\alpha^2)^{1/2}} = c \cos \phi\]

Combining this with Eq. A9.26a, it is easy to show
\[k = \frac{3c}{(9 + 12 \tan^2 \phi)^{1/2}}\]

(A9.26b)

Equations A9.26a and A9.26b are the original relations derived by Drucker and Prager (1952). To summarize the results:

Plane strain:
\[\alpha_{PS} = \frac{\tan \phi}{(9 + 12 \tan^2 \phi)^{1/2}}; \quad k_{PS} = \frac{3c}{(9 + 12 \tan^2 \phi)^{1/2}}\]

(A9.27)

Three dimensions:
\[\alpha_{c,3D} = \frac{2 \sin \phi}{\sqrt{3(3 - \sin \phi)}}; \quad k_{c,3D} = \frac{6c \cos \phi}{\sqrt{3(3 - \sin \phi)}}\]

(A9.28a)

\[\alpha_{e,3D} = \frac{2 \sin \phi}{\sqrt{3(3 + \sin \phi)}}; \quad k_{e,3D} = \frac{6c \cos \phi}{\sqrt{3(3 + \sin \phi)}}\]

(A9.28b)

where \(\alpha_{c,3D}\) and \(k_{c,3D}\) are parameters in triaxial compression and \(\alpha_{e,3D}\) and \(k_{e,3D}\) are parameters in triaxial extension for use in 3D analyses.

For example, for \(\phi = 30^\circ\), \(\alpha_{PS} = 0.16\), \(k_{PS} = 0.83c\), \(\alpha_{c,3D} = 0.23\), \(k_{c,3D} = 1.2c\), \(\alpha_{e,3D} = 0.16\) and \(k_{e,3D} = 186c\). Again, it is seen that the values of the parameters in plane strain and three dimensions are very different from each other.

**Case B: Non-Associated Flow Rule: \(d_0 \neq 1\)**

For this case, squaring both sides of Eq. A9.25a, one gets the following equation for \(\alpha\):
\[a_1\alpha^4 + a_2\alpha^2 + a_3 = 0\]

(A9.29)

where
\[a_1 = 9d_0^2(3 + \sin^2 \phi) ; \quad a_2 = -(9 + 6d_0 \sin^2 \phi) \quad \text{and} \quad a_3 = \sin^2 \phi\]
Equation A9.29 can be solved to evaluate $\alpha$ for a given value of $\phi$. From Eq. A9.25b,

$$k = \frac{(1 - 3d_0\alpha^2)}{(1 - 3d_0^2\alpha^2)^{1/2}} c \cos \phi$$ (A9.30)

Once $\alpha$ is found from Eq. A9.29, the value is substituted into Eq. A9.30 to find the value for $k$.

For example, consider the case for $\phi = 30^\circ$.

For $d_0 = 1$: $a_1 = 29.25$, $a_2 = -10.5$ and $a_3 = 0.25 \Rightarrow \alpha = 0.1601$, which agrees with that calculated earlier using the analytical equations A9.27.

For $d_0 = 0.5$: $a_1 = 7.31$, $a_2 = -9.75$ and $a_3 = 0.25 \Rightarrow \alpha = 0.1617$, which is very close to that calculated with $d_0 = 1$

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**Appendix 10. Exact Elastic Predictor for Exponential Elasticity**

Splitting elastic constitutive relations into volumetric and deviatoric parts

$$\dot{p} = K\dot{\varepsilon}_{\text{v}}$$ (A10.1a)

and

$$\dot{s}_{ij} = 2G\dot{\varepsilon}_{\text{d}}$$ (A10.1b)

where $\dot{s}_{ij} = \dot{\sigma}_{ij} - \dot{p}\delta_{ij}$ (A10.2a)

and

$$\dot{\varepsilon}_{ij} = \dot{\varepsilon}_{ij} - \frac{1}{3}\dot{\varepsilon}_{kk}\delta_{ij}$$ (A10.2b)

Using the relationships between $K$ and $G$, and $p$:

$$G = \tilde{G}_0(p + p_r)^{\nu_v}; \quad K = \widetilde{K}_0(p + p_r)^{\nu_v}$$ (A10.3a)

where $\tilde{G}_0 = G_0 p_a \left(\frac{2.97 - e_0}{1 + e_0}\right)^2$ and $\widetilde{K}_0 = \frac{2(1 + \nu)}{3(1 - 2\nu)} \tilde{G}_0$ (A10.3b)

Let $\Delta \varepsilon_{ij} = x\Delta \varepsilon_{ij}, x \in [0,1]$ (A10.4)

with $x = 0$ at the beginning of the increment, $x = 1$ at the end of the increment and $\Delta \varepsilon_{ij}$ is the total strain increment, which is a constant.

Combining Eqs. A10.1a and A10.3a and integrating the rate equations
\[
\int_{p_n}^{p} \frac{\dot{p}}{(p + p_t)^{n_e}} = K_0 \int_{0}^{\varepsilon_{kk}} \varepsilon_{kk}
\]

\[
(p + p_t)^{1-n_e} - (p_n + p_t)^{1-n_e} = K_0 (1-n_e) \delta \varepsilon_{pp}
\]

\[
p = \left[ (p_n + p_t)^{1-n_e} + K_0 (1-n_e) \delta \varepsilon_{pp} \right]^{1/(1-n_e)} - p_t
\]

\[
p = \left[ (p_n + p_t)^{1-n_e} + K_0 (1-n_e) x \Delta \varepsilon_{pp} \right]^{1/(1-n_e)} - p_t
\]  
\[
(A10.5)
\]

When \( x = 1 \), the above equation gives the elastic predictor ("trial" stress) as

\[
p^r = \left[ (p_n + p_t)^{1-n_e} + K_0 (1-n_e) \Delta \varepsilon_{pp} \right]^{1/(1-n_e)} - p_t
\]  
\[
(A10.6)
\]

Noting that \( \delta \varepsilon_{ij} = x \Delta \varepsilon_{ij} \), and

\[
\dot{\varepsilon}_{ij} = \dot{x} \Delta \varepsilon_{ij}
\]

Eq. A.8.32b is written as

\[
\dot{s}_{ij} = 2 G \dot{\varepsilon}_{ij} = 2 \tilde{G}_0 (p + p_t)^{n_e} \Delta \varepsilon_{ij} \dot{x}
\]

\[
\dot{s}_{ij} = 2 G \dot{\varepsilon}_{ij} = 2 \tilde{G}_0 \left[ (p_n + p_t)^{1-n_e} + K_0 (1-n_e) x \Delta \varepsilon_{pp} \right]^{n_e/(1-n_e)} \Delta \varepsilon_{ij} \dot{x}
\]

The elastic predictor is computed as

\[
\int_{s_{ij}^0}^{s_{ij}^r} = \int_{0}^{1} 2 \tilde{G}_0 \left[ (p_n + p_t)^{1-n_e} + K_0 (1-n_e) x \Delta \varepsilon_{pp} \right]^{n_e/(1-n_e)} \Delta \varepsilon_{ij} \dot{x}
\]

\[
s_{ij}^r = s_{ij}^n + 2 \tilde{G}_0 \Delta \varepsilon_{ij} ; \quad G_{ave} = \frac{\tilde{G}_0}{K_0 \Delta \varepsilon_{pp}} (p^r - p_n)
\]  
\[
(A10.7)
\]

The stress tensor is obtained as

\[
\sigma_{ij}^r = s_{ij}^r + p^r \delta_{ij}
\]  
\[
(A10.8)
\]

Note that when \( \Delta \varepsilon_{pp} = 0 \), Eq. A10.7 is not valid since \( G_{ave} \) goes to \( \infty \). Let’s find the limit for the case of \( n_e = 0.5 \).

\[
p^r = \left[ (p_n + p_t)^{0.5} + 0.5 \tilde{K}_0 \Delta \varepsilon_{pp} \right]^2 - p_t
\]

\[
= p_n + p_t + \tilde{K}_0 (p_n + p_t)^{0.5} \Delta \varepsilon_{pp} + 0.25 \tilde{K}_0^2 \Delta \varepsilon_{pp}^2 - p_t
\]

As \( \Delta \varepsilon_{pp} \to 0 \):

\[
p^r \to p_n + \tilde{K}_0 (p_n + p_t)^{0.5} \Delta \varepsilon_{pp}
\]

\[
G_{ave} = \frac{\tilde{G}_0}{K_0 \Delta \varepsilon_{pp}} \left[ \tilde{K}_0 (p_n + p_t)^{0.5} \Delta \varepsilon_{pp} \right] = \tilde{G}_0 (p_n + p_t)^{0.5} = G_n
\]

This must be recognized and dealt with in computer implementations. For example, it could be handled as follows:
$$G_{ave} = \frac{\bar{G}_0}{K_0\Delta\epsilon_{pp}} (p^r - p_n) \quad \text{for } \Delta\epsilon_{ek} \geq TOL$$

and

$$G_{ave} = \frac{\bar{G}_0}{K_0(TOL)} (\bar{p}^r - p_n) \quad \text{for } \Delta\epsilon_{kk} < TOL$$

where $\bar{p}^r = [(p_n + p_I)^{1-n_k} + K_0(1-n_c)(TOL)]^{1/(1-n_c)} - p_i$ and $TOL$ is a suitable small number.

Appendix 11: Computer Coding for Computing Incremental Quantities for Von Mises Model using $S$–Space Formulation (Section 11.3.2)

The formulation presented in section 11.3.2 uses nonlinear hardening and the “Exact” method of integrating the hardening variable.

Definition of variables:

Control parameters
- NPROP: Number of model parameters ($= 5$)
- NPIV: Number of PIVs ($= 6$)
- IELAST = 1: Find elastic stiffness, = 0: Find elasto-plastic stiffness
- ISTIFF = 1: Find stiffness, = 0: Do not find stiffness
- ISTRES = 1: Find incremental stress, = 0: Do not find incremental stress

Property array (brought in from the main program)
- PROP(1): $E$; PROP(2): $\nu$; PROP(3): $m$; PROP(4): $f_k$; PROP(5): $k_0$

State variable array (brought in from the main program)
- STOR(1): $\Delta\lambda$; STOR(2): $L$ (loading index); STOR(3): $k_n$; STOR(4): $k_{n+1}$
- STOR(5): $\varepsilon^p_n$; STOR(6): $\varepsilon^p_{n+1}$

Properties
- XM: $m$; XKF: $f_k$; XK0: $k_0$; XE: $E$; XNU: $\nu$; XK: $K_{n+1}$; XG: $G_{n+1}$

Hardening variable
- YSB: $k_n$; YSF: $k_{n+1}$; ZETAB: $\varepsilon^p_n$; ZETAF: $\varepsilon^p_{n+1}$

Stresses
- SIGB(6): $\sigma_n$; SIGBX(3,3): $\sigma^p_{ij}$; SIGF(6): $\sigma_{n+1}$; SIGFX(3,3): $\sigma^p_{ij}^{n+1}$
- DSIG(6): $\Delta\sigma$; DSIGX(3,3): $\Delta\sigma_{ij}$
- SB(3,3): $s^a_{ij}$; SF(3,3): $s_{ij}^{n+1}$; STRI(3,3): $s_{ij}^{pr}$; RJB: $J_n$; RJF: $J_{n+1}$; RJTRI: $J^{tr}$
- RIB: $I_n$; RIF: $I_{n+1}$; RITRI: $I^{tr}$

Strains
- DEP(6): $\Delta\epsilon$; DEPX(3,3): $\Delta\epsilon_{ij}$

Other
- DEL(3,3): $\delta_{ij}$
Cistique = 0 and ISTRE = 1: Begin Calculations to find incremental quantities
Initialize XM, XKF, XK0, XE, XNU
Initialize YSB, YSF=YSB, ZETAB, ZETAF=ZETAB
Initialize SIGB(6), SIGBX(3,3)
Initialize DEP(6), DEPX(3,3) (note: DEPX(1,2)=DEP(4)/2, etc.)
Initialize DEL(3,3)
XK=XE/(3.0*(1-2*XNU)), XG=XE/(2.0*(1+XNU))
Calculate: RIB, SB(3,3)
Calculate RITRI, STRI(3,3), RJTRI
Set SIGFX(3,3)=STRI(3,3)+RITRI*DEL(3,3)/3.0 and define SIGF(6)
FTRI=2.0*RJTRI**2-2*YSF**2/3
C Decide if the strain increment causes elastic or elasto-plastic response
IF(FTRI.LE.0.0) THEN
C Elastic
DSIG(6)=SIGF(6)-SIGB(6)
STOR(3)=0.0
RETURN TO THE MAIN PROGRAM
ENDIF
C Elasto-Plastic
Calculate B1, B2 and B3
Solve for DLAM = Δλ (choose positive of the roots)
Calculate X0 = x0
SF(3,3) = STRI(3,3)/X0
SIGFX(3,3)=SF(3,3)+RITRI*DEL(3,3)/3
Calculate DSIG(6)
ZETAF=ZETAB+2*DLAM*RJTRI/X0
Calculate constants C1, A2 and B
YSF=XK0+C1*ZETAF/(A2+B*ZETAF)
STOR(1)=DLAM
STOR(4)=YSF
STOR(6)=ZETAF
RETURN TO THE MAIN PROGRAM
Cistique = 1 and ISTRE = 0: Begin Calculations to find the tangent stiffness
C operator
C Determine if elastic or elasto-plastic stiffness is required
IF(IELAST=1 OR STOR(2) ≤ 0) THEN
C Find elastic stiffness
Calculate elastic stiffness C(6,6)
RETURN TO THE MAIN PROGRAM
ENDIF
C
C Find elasto-plastic consistent
Initialize XM, XKF, XK0, XE, XNU
Initialize YSF, ZETAF
Initialize DLAM
Initialize SIGB(6), SIGBX(3,3)
Initialize SIGF(6), SIGFX(3,3)
Calculate SF(3,3), RIF, RIF
Initialize DEP(6), DEPX(3,3) (note: DEPX(1,2)=DEP(4)/2, etc.)
Initialize DEL(3,3)
XK=XE/(3.0*(1-2*XNU)), XG=XE/(2.0*(1+XNU))
Calculate: RIB, SB(3,3)
Calculate RITRI, STRI(3,3), RJTRI
X0=1.0+4.0*XG*DLAM
C
Calculate B1, B2, B3, B1BAR, B2BAR, B3BAR, F1BAR
F1=4.0*XG*F1BAR/(X0**2)
C Calculate consistent operator
C The following code is in Fortran; modify this to suit the language of your choice
C Define an array II(6) as
DATA II/11,22,33,12,23,13/
X1=XK-2.0*XG/(3.0*X0)
X2=XG/X0
DO 100 M=1,6
  I=II(M)/10
  J=MOD(II(M),10)
  DO 100 N=1,6
    K=II(N)/10
    L=MOD(II(N),10)
    C(M,N)=X1*DEL(I,J)*DEL(K,L)
    C(M,N)=C(M,N)+X2*(DEL(I,K)*DEL(J,L)+DEL(I,L)*DEL(J,K))
    C(M,N)=C(M,N)-F1*SF(I,J)*SF(K,L)
100 CONTINUE
C   RETURN TO THE MAIN PROGRAM
C============================================================================