4
Linear Elastic Constitutive Relations

Introduction

The number of equations available through the principles of continuum mechanics (conservation of mass and energy and principles of linear and angular momentums) is inadequate to solve boundary value problems. The additional required equations are provided through what are known as the constitutive relations. In this chapter, we will discuss the details of the linear elastic constitutive relations.

4.1. Fundamentals

An elastic material is defined as one where the stress $\sigma$ depends only on the strain $\varepsilon$, not on the past thermodynamic history (Eringen, 1967). Limiting our discussion to processes that are adiabatic (where the heat loss or gain is absent) and isothermal (where the temperature remains constant), the laws of thermodynamics lead to the following result:

$$\sigma_{ij} = \frac{\partial u}{\partial \varepsilon_{ij}}; \quad u = u(\varepsilon)$$ (4.1)

where $\sigma_{ij}$ and $\varepsilon_{ij}$ are the stress and strain tensors respectively, and $u$ is the strain energy density (energy per unit volume). The existence of $u$ is established by using the laws of thermodynamics as well. The derivation is beyond the scope of this book; the reader is referred to any book on continuum mechanics (e.g., Mase, 1970; Eringen, 1967).

The symmetry of the stress tensor is assured by

$$\sigma_{ij} = \frac{1}{2} \left( \frac{\partial u}{\partial \varepsilon_{ij}} + \frac{\partial u}{\partial \varepsilon_{ji}} \right)$$ (4.2)

Noting that $u$ is a function of strain $\varepsilon$, it follows that Eq. 4.2. is a constitutive law for elastic solids when thermal effects are neglected. The models represented by Eq. 4.2. is called the Green-elastic or hyperelastic models. It may be noted that Eq. 4.2. is valid for isotropic as well as anisotropic solids, and linear as well as nonlinear elastic solids. On the other hand, in Cauchy elastic models, the stress is expressed as a direct function of strain as $\sigma = \phi(\varepsilon)$

We will limit the discussion in this chapter to Green-elastic models.

Let’s use a polynomial expression for $u$ as

$$u = A + B_{kl} \varepsilon_{kl} + \frac{1}{2} C_{kmn} \varepsilon_{kl} \varepsilon_{mn} + ....$$ (4.3a)
We require that \( u \) be invariant under coordinate axis rotations. Then, based on the coordinate transformation rules for second order tensors (strain, in this case, which transforms as \( \varepsilon' = a^T \varepsilon a \) or \( \varepsilon'_{kl} = d_{kl}\varepsilon_{ij}a_{ij} \)):

\[
\begin{align*}
u' &= A' + B'_{kl}\varepsilon'_{kl} + \frac{1}{2}C'_{klmn}\varepsilon'_{kl}\varepsilon'_{mn} + \ldots \\
&= A' + B'_{kl}a_{kl}a_{ij}\varepsilon_{ij} + \frac{1}{2}C'_{klmn}(a_{kl}a_{ij}\varepsilon_{ij})(a_{mp}a_{nq}\varepsilon_{pq}) + \ldots \\
&= A' + (B'_{kl}a_{kl}a_{ij})\varepsilon_{ij} + \frac{1}{2}(C'_{klmn}a_{kl}a_{ij}a_{mp}a_{nq})\varepsilon_{ij}\varepsilon_{pq} + \ldots \\
\end{align*}
\]

(4.3b)

where \( a \) is the transformation matrix appearing in the transformation of vectors \( x' = ax \) (Chapter 2). Noting \( uu' = \), comparison of Eqs. 4.3a and 4.3b reveals that

\[
A = A', \quad B_{ij} = B'_{kl}a_{kl}a_{ij}, \quad C_{ijpq} = C'_{klmn}a_{kl}a_{ij}a_{mp}a_{nq} \quad (4.3c)
\]

The constants of the polynomial follow the transformation rules for tensors, and hence are tensors.

Now, taking \( u = 0 \) in Eq. 4.3a when \( \varepsilon_{ij} = 0 \), one gets \( A = 0 \). Since we are only interested in linear elasticity in this chapter, we only need to retain up to the quadratic term in Eq. 4.3a as

\[
u = B_{kl}\varepsilon_{kl} + \frac{1}{2}C_{klmn}\varepsilon_{kl}\varepsilon_{mn} \quad (4.3b)
\]

Differentiating \( u \) with respect to strain

\[
\frac{\partial u}{\partial \varepsilon_{ij}} = B_{kl}\delta_{kl}\delta_{ij} + \frac{1}{2}C_{klmn}\delta_{kl}\delta_{ij}\varepsilon_{mn} + \frac{1}{2}C_{klmn}\varepsilon_{kl}\delta_{mn}\delta_{ij} \\
= B_{ij} + \frac{1}{2}(C_{ijnm}\varepsilon_{mn} + C_{kij}\varepsilon_{kl}) \quad (4.4)
\]

Substituting into Eq. 4.2

\[
\sigma_{ij} = \frac{1}{2}(B_{ij} + B_{ji}) + \frac{1}{4}(C_{ijmn}\varepsilon_{mn} + C_{kij}\varepsilon_{kl} + C_{jimn}\varepsilon_{mn} + C_{kiij}\varepsilon_{kl}) \quad (4.5a)
\]

Assuming that \( \sigma_{ij} = 0 \) for \( \varepsilon_{ij} = 0 \), one has \( B_{ij} = B_{ji} = 0 \).

From 4.3b:

\[
\frac{\partial^2 u}{\partial \varepsilon_{ij}\varepsilon_{kl}} = C_{ijkl}, \quad \frac{\partial^2 u}{\partial \varepsilon_{kl}\varepsilon_{ij}} = C_{klij} \\
\frac{\partial^2 u}{\partial \varepsilon_{ij}\varepsilon_{ij}} = \frac{\partial^2 u}{\partial \varepsilon_{kl}\varepsilon_{kl}} \Rightarrow C_{ijkl} = C_{klij} \quad (4.5b)
\]

Hence \( C_{ijkl} \) is symmetric both in \( i \) and \( k \) and in \( j \) and \( \ell \). Thus, in Eq. 4.5a,
\[ C_{ijmn} = C_{mnij} \quad (4.6) \]

Hence,
\[
\sigma_{ij} = \frac{1}{4} \left( C_{ijmn} \varepsilon_{mn} + C_{ijkl} \varepsilon_{kl} + C_{jimn} \varepsilon_{mn} + C_{jikl} \varepsilon_{kl} \right)
= \frac{1}{2} \left( C_{ijmn} \varepsilon_{mn} \right) + \frac{1}{2} \left( C_{jimn} \varepsilon_{mn} \right)
= \frac{1}{2} \left( C_{ijmn} + C_{jimn} \right) \varepsilon_{mn} \quad (4.7)
\]

Now from the symmetry of the stress and strain tensors:
\[ C_{ijmn} = C_{jimn} = C_{ijmn} \quad (4.8a) \]

Then
\[ \sigma_{ij} = C_{ijkl} \varepsilon_{kl} \quad (4.8b) \]

with \( C_{ijkl} \) possessing the symmetries stated in Eq. 4.6. and Eq. 4.8a.

Using the Voigt (1925) notation, let’s arrange the six independent components of \( \sigma_{ij} \) and \( \varepsilon_{ij} \) in vectors as
\[
\begin{bmatrix}
\sigma_{11} \\
\sigma_{22} \\
\sigma_{33} \\
\sigma_{12} \\
\sigma_{23} \\
\sigma_{13}
\end{bmatrix}
\quad \begin{bmatrix}
\varepsilon_{11} \\
\varepsilon_{22} \\
\varepsilon_{33} \\
\gamma_{12} \\
\gamma_{23} \\
\gamma_{13}
\end{bmatrix}
\quad (4.9a)
\]
\[
\begin{bmatrix}
\varepsilon_{11} \\
\varepsilon_{22} \\
\varepsilon_{33} \\
\gamma_{12} \\
\gamma_{23} \\
\gamma_{13}
\end{bmatrix}
\quad \begin{bmatrix}
\sigma_{11} \\
\sigma_{22} \\
\sigma_{33} \\
\sigma_{12} \\
\sigma_{23} \\
\sigma_{13}
\end{bmatrix}
\quad (4.9b)
\]

where \( \gamma_{12} = \frac{1}{2} (\varepsilon_{12} + \varepsilon_{21}) \), \( \gamma_{23} = \frac{1}{2} (\varepsilon_{23} + \varepsilon_{32}) \) and \( \gamma_{13} = \frac{1}{2} (\varepsilon_{13} + \varepsilon_{31}) \). We thus use the engineering shear strains \( \gamma \)’s instead of the tensor shear strains \( \varepsilon_{12} \) etc. to be compatible with the relation needed in the finite element analysis described in a subsequent chapter.

In the matrix-vector form, Eq. 4.8b may then be written as
\[
\begin{bmatrix}
\sigma_{11} \\
\sigma_{22} \\
\sigma_{33} \\
\sigma_{12} \\
\sigma_{23} \\
\sigma_{13}
\end{bmatrix}
= \begin{bmatrix}
C_{1111} & C_{1122} & C_{1133} & C_{1112} & C_{1123} & C_{1131} \\
C_{2211} & C_{2222} & C_{2233} & C_{2212} & C_{2223} & C_{2231} \\
C_{3311} & C_{3322} & C_{3333} & C_{3312} & C_{3323} & C_{3331} \\
C_{1211} & C_{1222} & C_{1233} & C_{1212} & C_{1223} & C_{1231} \\
C_{2311} & C_{2322} & C_{2333} & C_{2312} & C_{2323} & C_{2331} \\
C_{1311} & C_{1322} & C_{1333} & C_{1312} & C_{1323} & C_{1331}
\end{bmatrix}
\begin{bmatrix}
\varepsilon_{11} \\
\varepsilon_{22} \\
\varepsilon_{33} \\
\gamma_{12} \\
\gamma_{23} \\
\gamma_{13}
\end{bmatrix} \quad (4.10a)
\]
\[ \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ C_{21} & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\ C_{31} & C_{32} & C_{33} & C_{34} & C_{35} & C_{36} \\ C_{41} & C_{42} & C_{43} & C_{44} & C_{45} & C_{46} \\ C_{51} & C_{52} & C_{53} & C_{54} & C_{55} & C_{56} \\ C_{61} & C_{62} & C_{63} & C_{64} & C_{65} & C_{66} \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \gamma_{12} \\ \gamma_{23} \\ \gamma_{13} \end{bmatrix} = (4.10b) \]

\[ \bar{\sigma} = \bar{C}\bar{\varepsilon} \quad (4.10c) \]

where the symmetry with respect to the last two subscripts has already been utilized in simplifying as \( \frac{1}{2}(C_{1112} + C_{1121}) = C_{1112} \), etc. Thus, there are at most 36 constants in Eq. 4.10. However, now making use of the symmetry \( C_{ijmn} = C_{nmij} \) (Eq. 4.6), it is noted that \( \bar{C} \) is symmetric. Hence there are only 21 independent constants in Eq. 4.10b.

Remark
We recall Hooke’s law for an isotropic, elastic solid in uniaxial loading as
\[ \sigma = E\varepsilon \quad (4.11) \]
where \( \sigma \) and \( \varepsilon \) are the uniaxial stress and strain and \( E \) is the Young’s modulus. Extending this to the multi-axial case, we define the generalized Hooke’s law as
\[ \sigma_{ij} = C_{ijkl}\varepsilon_{kl} \quad (4.12) \]
where \( C_{ijkl} \) is the 4th order elastic material tensor. Before the consideration of symmetries, it is easy to see that 81 constants are needed to define \( C_{ijkl} \) (a quantity that relates the 9 components of \( \sigma_{ij} \) and 9 components of \( \varepsilon_{ij} \) involves 81 constants). Now the symmetry of \( \sigma_{ij} \) and \( \varepsilon_{ij} \) reduces the number of constants to 36 (= 6 \times 6). Further, we saw above that the use of Green-elasticity (Eq. 4.2) reduces the number of constants from 36 to 21. Further reductions are possible based on certain other material symmetries, which is discussed in the subsequent sections.
4.2. Linear-Elastic Relations for Anisotropic Solids with Symmetries

The general linear elastic stress-strain relation for an anisotropic solid is given by Eq. 4.10, which requires the specification of 21 constants (when Green-elasticity is used). However, most materials possess certain material symmetries that can be exploited to further reduce the number of independent constants needed for the relation in Eq. 4.10.

![Figure 1 Specimens Cut from Isotropic Material](image)

Figure 1 Specimens Cut from Isotropic Material

To understand material symmetries, first consider an isotropic solid shown in Fig. 1. The microstructure of the material has random orientation at the appropriate scale to render the elastic properties directionally independent. (Note that a metal may be anisotropic at a nano scale due to anisotropic orientation of atoms within crystals, it may still be isotropic at the macroscopic due to the random orientation of grains.)

Let us call the coordinate system with respect to which the loading is applied as “loading directions.” For an isotropic material, when the loading directions are rotated from $x_1 - x_2 - x_3$ to $x'_1 - x'_2 - x'_3$, the stress-strain behavior will remain unchanged.

For example, suppose that when a specimen is subjected to a principal strain loading $(0.01, 0.02, 0.03)$ % in the loading directions $x_1 - x_2 - x_3$, the stresses developed in these directions (i.e., the $x_1 - x_2 - x_3$ directions) are $(10, 20, 30)$ MPa. Then when another specimen cut from the same parent material is subjected to the same principal strain loading in the rotated directions $x'_1 - x'_2 - x'_3$, the developed stresses must be $(10, 20, 30)$ MPa in the $x'_1 - x'_2 - x'_3$ directions.
This will not be the case when the material has different properties in different directions. However, when certain symmetries exist, the behavior in some specific loading directions remains unchanged. Three such symmetries are considered in this section (Figs. 2, 3 and 4).

Figure 2 Reflection about the $x_{30} = 0$ Plane

In Fig. 2, we have a material for which the behavior of specimens A and B are identical, where B is obtained as a mirror-image (or reflection) of A about the $x_{30} = 0$ plane. Examples of materials that possess such symmetry are monoclinic crystals (gypsum, talc, etc.) and a rock mass with a set of parallel fractures arranged in a certain manner. The behavior will be identical with respect to the $x_1 - x_2 - x_3$ and $x_1' - x_2' - x_3'$ coordinate systems. The material is said to have one plane of symmetry - the $x_3$-plane (i.e., the plane normal to the $x_3$-axis). Such materials are called monoclinic materials.
In Fig. 3, we have a material that has three orthogonal planes of symmetry; the planes of symmetry are the $x_1-$, $x_2-$ and $x_3-$ planes. Such materials are called the orthotropic materials. Isotropic materials reinforced with cylindrical fibers as shown in Fig. 3 are examples of orthotropic materials.

In Fig. 4, we have a material for which the properties on the $x_3 = 0$ plane are directionally independent. Such materials are called the transversely isotropic materials. The axis of symmetry is the $x_3-$ axis. Natural soils are transversely isotropic with the axis of symmetry being the original depositional direction, usually vertical.

The material symmetries reduce the number of constants from 21. The reduction is determined as follows. Let the stress-strain relation in the $x_1-x_2-x_3$ and $x'_1-x'_2-x'_3$ systems be

$$\sigma_{ij} = C_{ijkl} \epsilon_{kl} \quad (4.13a)$$
and

$$\sigma'_{ij} = C'_{ijkl} \epsilon'_{kl} \quad (4.13b)$$

respectively. Recall the discussion we had in the case of the isotropic material. Based on this, when the symmetry dictates that the behavior be the same in the $x_1-x_2-x_3$ and $x'_1-x'_2-x'_3$ directions, we require that $\sigma'_{ij} = \sigma_{ij}$ when $\epsilon'_{ij} = \epsilon_{ij}$ and vise-versa. This requires

$$C'_{ijkl} = C_{ijkl} \quad (4.14)$$

Based on Eq. 4.3c, Eq. 4.14 becomes
Now we apply Eq. 4.15 for each of the three symmetries discussed earlier and determine the reduction in the number of constants in the following sections.

4.2.1. Monoclinic Materials (One Plane of Symmetry, Fig. 2)

Referring to Fig. 2, the transformation matrix appearing in $x' = ax$ is (Chapter 2):

$$a = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

Noting the zeros in $a$, let's evaluate two typical constants in $\bar{C}$ (Eq. 4.10):

$$\bar{C}_{44}'' = C_{1212} = C_{pqrs} a_p a_q a_r a_s = C_{1212} a_{1212} a_{1212} a_{1212} a_{1212} = C_{1212} \Rightarrow C_{1212} \neq 0$$

That is, $C_{1212}$ is not necessarily zero. It may be noted that a similar result is obtained when $\text{sign}(a_p) \times \text{sign}(a_q) \times \text{sign}(a_r) \times \text{sign}(a_s) = 1$

This leads to the conclusion that the following 13 constants are not necessarily zero (considering only the upper diagonal coefficients in Eq. 4.10a):

$$C_{1111}, C_{1122}, C_{1133}, C_{1112}, C_{2222}, C_{2233}, C_{2212}, C_{3333}, C_{3312}, C_{1212}, C_{2323}, C_{2313}, C_{1313}$$

Now consider:

$$\bar{C}_{15}' = C_{1123} = C_{pqrs} a_p a_q a_r a_s = C_{1123} a_{1123} a_{1123} a_{1123} = -C_{1123} \Rightarrow C_{1123} = 0$$

Noting that this occurs when $\text{sign}(a_p) \times \text{sign}(a_q) \times \text{sign}(a_r) \times \text{sign}(a_s) = -1$

the following 8 constants (among the upper diagonal coefficients) are found to fall in this group:

$$C_{1123}, C_{1113}, C_{2223}, C_{2213}, C_{3323}, C_{3313}, C_{1223}, C_{1213}$$

The matrix in Eq. 4.10a now becomes:

$$\bar{C} = \begin{bmatrix}
C_{1111} & C_{1122} & C_{1133} & C_{1112} & 0 & 0 \\
C_{1122} & C_{2222} & C_{2233} & C_{2212} & 0 & 0 \\
C_{1133} & C_{2233} & C_{3333} & C_{3312} & 0 & 0 \\
C_{1112} & C_{2212} & C_{3312} & C_{1212} & 0 & 0 \\
0 & 0 & 0 & 0 & C_{2323} & C_{2313} \\
0 & 0 & 0 & 0 & C_{2313} & C_{1313}
\end{bmatrix} \quad (4.16)$$
The number of independent constants is 13.

![Figure 5](image)

Figure 5 Reflection about the $x_2 = 0$ Plane

### 4.2.2. Orthotropic Materials (Fig. 3)

The symmetry shown in Fig. 2, which represents reflection about the $x_3 = 0$ plane, gave us the matrix in Eq. 4.16. In addition to this, we now use the symmetry represented by the $x'_1 - x'_2 - x'_3$ axes shown in Fig. 5, where the $x'_1 - x'_2 - x'_3$ axes are obtained by reflecting the $x_1 - x_2 - x_3$ axes about the $x_2 = 0$ plane. It turns out that the number of independent constants obtained using two orthogonal planes of symmetry is the same as that using three orthogonal planes of symmetry, implying that the 3rd symmetry is dependent on the first two.

The transformation matrix is:

$$
\begin{bmatrix}
1 & 0 & 0 \\
0 & -1 & 0 \\
0 & 0 & 1
\end{bmatrix}
$$

We will start with Eq. 4.16 and see how the matrix $\bar{\mathbf{C}}$ further simplifies. Let’s evaluate a typical constant $C_{1122}$:

$$
\bar{C}_{12} = C_{1122} = C_{pqrs} \alpha_i \alpha_j \alpha_k \alpha_s C_{1122} a_{11} a_{12} a_{22} = C_{1122} a_{12} a_{22} a_{11} = C_{1122} \Rightarrow C_{1122} \neq 0
$$

That is, $C_{1122}$ is not necessarily zero. A similar result is obtained when

$$
\text{sign}(a_{pi}) \times \text{sign}(a_{qj}) \times \text{sign}(a_{sk}) \times \text{sign}(a_{sp}) = 1
$$

This leads to the conclusion that the following 9 constants (among the upper diagonal coefficients) are not necessarily zero:

$$
C_{1111}; C_{1122}; C_{1133}; C_{2222}; C_{2233}; C_{3333}; C_{1212}; C_{2323}; C_{1313}
$$

Now consider:

$$
\bar{C}_{14} = C_{1112} = C_{pqrs} \alpha_i \alpha_j \alpha_k \alpha_s C_{1112} a_{11} a_{12} a_{22} = -C_{1112} a_{12} a_{22} a_{11} = -C_{1112} \Rightarrow C_{1112} = 0
$$
Noting that this occurs when
\[ \text{sign}(a_{pl}) \times \text{sign}(a_{qr}) \times \text{sign}(a_{qr}) \times \text{sign}(a_{st}) = -1 \]
the following 4 constants are found to fall in this group:
\( C_{1112}; C_{2212}; C_{3312}; C_{2313} \)

The matrix \( \overline{C} \) (Eq. 4.16) now becomes:
\[
\overline{C} = 
\begin{bmatrix}
C_{1111} & C_{1122} & C_{1133} & 0 & 0 & 0 \\
C_{1122} & C_{2222} & C_{2233} & 0 & 0 & 0 \\
C_{1133} & C_{2233} & C_{3333} & 0 & 0 & 0 \\
0 & 0 & 0 & C_{1212} & 0 & 0 \\
0 & 0 & 0 & 0 & C_{2323} & 0 \\
0 & 0 & 0 & 0 & 0 & C_{3213}
\end{bmatrix}
\]  
(Eq. 4.17)

The number of independent constants is 9.

The reader may now verify that the reflection about the \( x_i = 0 \) plane, which corresponds to the transformation matrix
\[
\mathbf{a} = 
\begin{bmatrix}
-1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\]
leaves Eq. 4.17 unchanged (Problem 4.2).

It is easy to see by writing down six equations from Eq. 4.10 with the \( \overline{C} \) given by 4.17 that the normal stress-strain relations are uncoupled from the shear stress-strain relations; i.e., a shear strain will not produce a normal stress and a normal strain will not produce a shear stress. Now Eq. 4.17 may be inverted to find the inverse relation, which is of the form:
\[
\begin{bmatrix}
\varepsilon_{11} \\
\varepsilon_{22} \\
\varepsilon_{33} \\
\gamma_{12} \\
\gamma_{23} \\
\gamma_{13}
\end{bmatrix} = 
\begin{bmatrix}
\overline{S}_{11} & \overline{S}_{12} & \overline{S}_{13} & 0 & 0 & 0 \\
\overline{S}_{12} & \overline{S}_{22} & \overline{S}_{23} & 0 & 0 & 0 \\
\overline{S}_{13} & \overline{S}_{23} & \overline{S}_{33} & 0 & 0 & 0 \\
0 & 0 & 0 & \overline{S}_{44} & 0 & 0 \\
0 & 0 & 0 & 0 & \overline{S}_{55} & 0 \\
0 & 0 & 0 & 0 & 0 & \overline{S}_{66}
\end{bmatrix}
\begin{bmatrix}
\sigma_{11} \\
\sigma_{22} \\
\sigma_{33} \\
\sigma_{12} \\
\sigma_{23} \\
\sigma_{13}
\end{bmatrix}
\]
(Eq. 4.18a)

\[
\varepsilon = \overline{S} \sigma
\]
(Eq. 4.18b)

where \( \overline{S} \) is the compliance matrix.
It is useful to interpret the coefficients in Eq. 4.18a in terms of the familiar engineering constants \( E \), \( \nu \), etc. Consider a uniaxial stress loading in the \( x_1 \)-direction. For this loading

\[
\begin{align*}
\sigma_{22} &= 0; \quad \sigma_{33} = 0 \\
\varepsilon_{11} &= S_{11} \sigma_{11}; \quad \varepsilon_{22} = S_{12} \sigma_{11} = \frac{S_{12}}{S_{11}} \varepsilon_{11}; \quad \varepsilon_{33} = S_{13} \sigma_{11} = \frac{S_{13}}{S_{11}} \varepsilon_{11}
\end{align*}
\]

Comparing these with the corresponding relations for an isotropic material (discussed in the next section)

\[
\begin{align*}
\varepsilon_{11} &= \frac{1}{E} \sigma_{11}; \quad \varepsilon_{22} = -\nu \varepsilon_{11}; \quad \varepsilon_{33} = -\nu \varepsilon_{11}
\end{align*}
\]

where \( E \) and \( \nu \) are Young’s modulus and Poisson’s ratio, the following orthotropic constants are introduced:

\[
\begin{align*}
\tilde{S}_{11} &= \frac{1}{E_1}; \quad \tilde{S}_{12} = -\nu_{12} \Rightarrow \tilde{S}_{12} = -\frac{\nu_{12}}{E_1}; \quad \tilde{S}_{13} = -\nu_{13} \Rightarrow \tilde{S}_{13} = -\frac{\nu_{13}}{E_1}
\end{align*}
\]

We follow the similar logic for the remaining constants in the normal stress-strain relations.

The orthotropic shear stress-strain relations are

\[
\begin{align*}
\gamma_{12} &= S_{44} \sigma_{12}; \quad \gamma_{23} = S_{55} \sigma_{23}; \quad \gamma_{13} = S_{66} \sigma_{13}
\end{align*}
\]

Comparing these with the corresponding relations for an isotropic material

\[
\begin{align*}
\gamma_{12} &= G \sigma_{12}; \quad \gamma_{23} = G \sigma_{23}; \quad \gamma_{13} = G \sigma_{13}
\end{align*}
\]

where \( G \) is the shear modulus, we introduce the following orthotropic shear moduli as

\[
\begin{align*}
S_{44} &= G_{12}; \quad S_{55} = G_{23}; \quad S_{66} = G_{13}
\end{align*}
\]

Thus, the nine (9) independent constants we have introduced are:

- The three Young’s moduli: \( E_1; \ E_2; \ E_3 \) \hspace{1cm} (4.19a)
- The three Poisson’s ratios: \( \nu_{12}; \ \nu_{23}; \ \nu_{13} \) \hspace{1cm} (4.19b)
- The three shear moduli: \( G_{12}; \ G_{23}; \ G_{13} \) \hspace{1cm} (4.19c)

In terms of these constants, the compliance matrix for an orthotropic material \( \mathbf{S} \) becomes

\[
\begin{bmatrix}
\varepsilon_{11} \\
\varepsilon_{22} \\
\varepsilon_{33} \\
\gamma_{12} \\
\gamma_{23} \\
\gamma_{13}
\end{bmatrix} =
\begin{bmatrix}
1 / E_1 & -\nu_{12} / E_2 & -\nu_{13} / E_3 & 0 & 0 & 0 \\
-\nu_{12} / E_1 & 1 / E_2 & -\nu_{23} / E_3 & 0 & 0 & 0 \\
-\nu_{13} / E_1 & -\nu_{23} / E_2 & 1 / E_3 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 / G_{12} & 0 & 0 \\
0 & 0 & 0 & 0 & 1 / G_{23} & 0 \\
0 & 0 & 0 & 0 & 0 & 1 / G_{13}
\end{bmatrix}
\begin{bmatrix}
\sigma_{11} \\
\sigma_{22} \\
\sigma_{33} \\
\sigma_{12} \\
\sigma_{23} \\
\sigma_{13}
\end{bmatrix}
\]

(4.20)

The \( \mathbf{S} \) matrix in Eq. 4.20 may be inverted to find the \( \mathbf{C} \) matrix. Skipping the details, the final result is
\[ \begin{align*}
\bar{C}_{11} &= (1 - \nu_{23}^2)/(E_2 E_3 d); \quad \bar{C}_{12} = (\nu_{12} + \nu_{13} \nu_{23})/(E_2 E_3 d); \quad \bar{C}_{13} = (\nu_{13} + \nu_{12} \nu_{23})/(E_2 E_3 d) \\
\bar{C}_{22} &= (1 - \nu_{13}^2)/(E_1 E_3 d); \quad \bar{C}_{23} = (\nu_{23} + \nu_{13} \nu_{12})/(E_1 E_3 d) \\
\bar{C}_{33} &= (1 - \nu_{12}^2)/(E_1 E_2 d)
\end{align*}\]  
\[ (4.21a) \]
\[ (4.21b) \]
\[ (4.21c) \]

where \( d \) is the determinant of the top-left 3 \times 3 sub-matrix in Eq. 4.20, given by:
\[ d = \frac{1}{E_1 E_2 E_3} \left(1 - \nu_{12}^2 - \nu_{23}^2 - \nu_{13}^2 - 2\nu_{12} \nu_{23} \nu_{13}\right) \]  
\[ (4.21d) \]

and \( \bar{C}_{44} = G_{12}; \quad \bar{C}_{55} = G_{23} \) and \( \bar{C}_{55} = G_{13}. \)  
\[ (4.21e) \]

### 4.2.3. Transversely Isotropic material (Fig. 4)

In this symmetry, the behavior must be the same for any value of \( \theta \) shown in Fig. 4. It turns out that by considering 3 different values for \( \theta \), we can derive the final matrix for this case. We will consider \( \theta = 180^0, 90^0 \) and \( 45^0 \) in this order.

For \( \theta = 180^0 \):
\[
a = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (4.22a)
\]

By following the same approach in previous sections (e.g., Section 4.2.2), it is easy to show
\[
\bar{C} = \begin{bmatrix}
C_{1111} & C_{1122} & C_{1133} & C_{1112} & 0 & 0 \\
C_{2211} & C_{2222} & C_{2233} & C_{2212} & 0 & 0 \\
C_{3311} & C_{3322} & C_{3333} & C_{3312} & 0 & 0 \\
C_{1211} & C_{1222} & C_{1233} & C_{1212} & 0 & 0 \\
0 & 0 & 0 & 0 & C_{2323} & C_{2313} \\
0 & 0 & 0 & 0 & C_{1323} & C_{1313}
\end{bmatrix} \quad (4.22b)
\]

For \( \theta = 90^0 \):
\[
a = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (4.23a)
\]

One gets:
\[
\mathbf{C} = \begin{bmatrix}
C_{111} & C_{112} & C_{113} & C_{112} & 0 & 0 \\
C_{112} & C_{111} & C_{113} & -C_{1112} & 0 & 0 \\
C_{113} & C_{113} & C_{333} & 0 & 0 & 0 \\
C_{1112} & -C_{1112} & 0 & C_{1212} & 0 & 0 \\
0 & 0 & 0 & 0 & C_{2323} & 0 \\
0 & 0 & 0 & 0 & 0 & C_{2323}
\end{bmatrix}
\]  
(4.23b)

For \( \theta = 45^0 \):
\[
\mathbf{a} = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ -1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}
\]  
(4.24a)

\( \mathbf{C} \) becomes
\[
\mathbf{\bar{C}} = \begin{bmatrix}
C_{111} & C_{112} & C_{113} & 0 & 0 & 0 \\
C_{112} & C_{111} & C_{113} & 0 & 0 & 0 \\
C_{113} & C_{113} & C_{333} & 0 & 0 & 0 \\
0 & 0 & 0 & (C_{111} - C_{112})/2 & 0 & 0 \\
0 & 0 & 0 & 0 & C_{2323} & 0 \\
0 & 0 & 0 & 0 & 0 & C_{2323}
\end{bmatrix}
\]  
(4.24b)

It may be verified that Eq. 4.24b remains valid for any other values of \( \theta \). The number of independent constants is 5.

\( \mathbf{\bar{C}} \) given in Eq. 4.24b may be inverted to obtain the corresponding compliance matrix, which will be in the following form:
\[
\mathbf{\bar{S}} = \begin{bmatrix}
S_{11} & S_{12} & S_{13} & 0 & 0 & 0 \\
S_{12} & S_{11} & S_{13} & 0 & 0 & 0 \\
S_{13} & S_{13} & S_{33} & 0 & 0 & 0 \\
0 & 0 & 0 & S_{44} & 0 & 0 \\
0 & 0 & 0 & 0 & S_{55} & 0 \\
0 & 0 & 0 & 0 & 0 & S_{66}
\end{bmatrix}
\]  
(4.25)

Note that the pattern seen in \( \mathbf{\bar{C}} \) (i.e., \( C_{13} = C_{23} \), etc.) is still preserved in \( \mathbf{\bar{S}} \). We will obtain a special relation for \( S_{44} \) later. First, following the same approach as for the orthotropic material (Section 4.2.2), we develop the following representation for \( \mathbf{\bar{S}} \) in terms of the familiar engineering constants:
By inverting this relationship we obtain the coefficients of $\mathbf{C}$. Noting that directions 1 and 2 are identical, Eq. 4.21 may be simplified to obtain:

$$
\begin{align*}
\epsilon_{11} &= \frac{1}{E_1} - \nu_{12}/E_4 - \nu_{13}/E_3 \quad 0 \quad 0 \quad 0 \\
\epsilon_{22} &= -\nu_{12}/E_4 \quad 1/E_4 - \nu_{13}/E_3 \quad 0 \quad 0 \quad 0 \\
\epsilon_{33} &= -\nu_{13}/E_4 - \nu_{13}/E_1 \quad 1/E_3 \quad 0 \quad 0 \quad 0 \\
\gamma_{12} &= 0 \quad 0 \quad 0 \quad 0 \quad 1/G_{12} \quad 0 \quad 0 \\
\gamma_{23} &= 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 1/G_{23} \quad 0 \\
\gamma_{13} &= 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 1/G_{23}
\end{align*}
$$

(4.26)

Using the relation seen in Eq. 4.24b that

$$
G_{12} = \frac{1}{E_1}(1 - \nu_{12}^2 - 2\nu_{13}^2 - 2\nu_{12}\nu_{13}) = \frac{1}{E_1}E_3(1 - \nu_{12}^2 - 2\nu_{13}^2)
$$

(4.27d)

and $\bar{C}_{11} = G_{12}$; $\bar{C}_{33} = G_{13}$ and $\bar{C}_{66} = G_{13}$.  (4.27e)

Using the relation seen in Eq. 4.24b that

$$
\bar{C}_{44} = \frac{(C_{1111} - C_{1122})}{2} = \frac{(\bar{C}_{11} - \bar{C}_{12})}{2}
$$

we get

$$
G_{12} = \frac{(1 - \nu_{12}^2 - 2\nu_{13}^2)}{2E_1E_3d} = \frac{E_1}{2(1 + \nu_{12})}
$$

(4.28a)

Hence $S_{44} = \frac{2(1 + \nu_{12})}{E_1}$

(4.28b)

### 4.3. Linear-Elastic Relations for Isotropic Solids

In isotropic solids, the material behavior is identical with respect to any arbitrary rotation of the coordinate system. Isotropy could be viewed as transverse isotropy with respect to two orthogonal axes.

We derived Eq. 4.24b be considering transverse isotropy with respect to symmetry about the $x_3$ axis (Fig. 4). By repeating the process with respect to one more axis of symmetry, it can be shown:
There are only two independent constants \( C_{1111} \) and \( C_{1122} \) in the above equation for \( \bar{C} \). \( \bar{C} \) may be inverted to find the compliance matrix \( \bar{S} \).

In items of the familiar engineering notations, the inverse relation is (obtained by simplifying 4.26):

\[
\begin{bmatrix}
\varepsilon_{11} \\
\varepsilon_{22} \\
\varepsilon_{33} \\
\gamma_{12} \\
\gamma_{23} \\
\gamma_{13}
\end{bmatrix}
= \begin{bmatrix}
\frac{1}{E} & -\nu/E & -\nu/E \\
-\nu/E & \frac{1}{E} & -\nu/E \\
-\nu/E & -\nu/E & \frac{1}{E} \\
0 & 0 & 1/G \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
\sigma_{11} \\
\sigma_{22} \\
\sigma_{33} \\
\sigma_{12} \\
\sigma_{23} \\
\sigma_{13}
\end{bmatrix}
\] (4.30a)

The coefficients of the stiffness matrix are (obtained by simplifying Eqs. 4.27 and 4.28):

\[
\bar{C} = \frac{E}{(1+\nu)(1-2\nu)}
\begin{bmatrix}
1-\nu & \nu & \nu & 0 & 0 & 0 \\
\nu & 1-\nu & \nu & 0 & 0 & 0 \\
\nu & \nu & 1-\nu & 0 & 0 & 0 \\
0 & 0 & 0 & (1-2\nu)/2 & 0 & 0 \\
0 & 0 & 0 & 0 & (1-2\nu)/2 & 0 \\
0 & 0 & 0 & 0 & 0 & (1-2\nu)/2
\end{bmatrix}
\] (4.30b)

Using Lame constants:

\[
C_{1122} = \lambda, \text{ and } \frac{1}{2}(C_{1111} - C_{1122}) = \mu
\] (4.30c)

where \( \lambda \) and \( \mu \) are the Lame constants,
\[
\bar{C} = \begin{bmatrix}
\lambda + 2\mu & \lambda & \lambda & 0 & 0 & 0 \\
\lambda & \lambda + 2\mu & \lambda & 0 & 0 & 0 \\
\lambda & \lambda & \lambda + 2\mu & 0 & 0 & 0 \\
0 & 0 & 0 & \mu & 0 & 0 \\
0 & 0 & 0 & 0 & \mu & 0 \\
0 & 0 & 0 & 0 & 0 & \mu
\end{bmatrix}
\] (4.30d)

The coefficients of \(\bar{C}\) may be generated using

\[C_{ijk\ell} = \lambda \delta_{ij} \delta_{k\ell} + \mu (\delta_{ik} \delta_{j\ell} + \delta_{i\ell} \delta_{jk})\] (4.31)

For example

\[
\bar{C}_{11} = C_{1111} = \lambda \delta_{11} \delta_{11} + \mu (\delta_{11} \delta_{11} + \delta_{11} \delta_{11}) = \lambda + 2\mu
\]

\[
\bar{C}_{13} = C_{1122} = \lambda \delta_{11} \delta_{22} + \mu (\delta_{12} \delta_{12} + \delta_{12} \delta_{12}) = \lambda
\]

\[
\bar{C}_{44} = (C_{1111} - C_{1122})/2 = \mu
\]

The stress-strain relationship can be split up into spherical and deviatoric and parts as follows:

\[
\sigma_{ij} = C_{ijk\ell} \varepsilon_{k\ell}
\]

\[
\sigma_{ij} = \left[\lambda \delta_{ij} \delta_{k\ell} + \mu (\delta_{ik} \delta_{j\ell} + \delta_{i\ell} \delta_{jk})\right] \varepsilon_{k\ell}
\]

\[
\sigma_{ii} = \sigma_{ij} \delta_{ij} = \left[\lambda \delta_{ij} \delta_{k\ell} \delta_{ij} + \mu (\delta_{ik} \delta_{j\ell} + \delta_{i\ell} \delta_{jk}) \delta_{ij}\right] \varepsilon_{k\ell}
\]

\[
= \left[3\lambda + 2\mu\right] \delta_{k\ell} \varepsilon_{k\ell}
\]

\[
= \left[3\lambda + 2\mu\right] \varepsilon_{kk}
\]

\[
\sigma_{ii} = 3 \left(\lambda + \frac{2}{3} \mu\right) \varepsilon_{kk} = 3K \varepsilon_{kk}; \quad K = \lambda + \frac{2}{3} \mu
\] (4.32)

Equation 4.32 is the spherical part of the stress-strain relation, which relates the mean normal pressure \(p = \sigma_{ii}/3\) to the volumetric strain \(\varepsilon_v = \varepsilon_{kk}\), where \(K\) is the bulk modulus.

\[
s_{ij} = \sigma_{ij} - \frac{\sigma_{kk}}{3} \delta_{ij}
\]

\[
= \left[\lambda \delta_{ij} \delta_{k\ell} + \mu (\delta_{ik} \delta_{j\ell} + \delta_{i\ell} \delta_{jk})\right] \varepsilon_{k\ell} - \left(\lambda + \frac{2}{3} \mu\right) \varepsilon_{kk} \delta_{ij}
\]

\[
= 2\mu \left\{\varepsilon_{ij} - \frac{1}{3} \varepsilon_{kk} \delta_{ij}\right\}
\]

\[
= 2\mu \varepsilon_{ij}
\] (4.33)
where \( e_{ij} \) is the deviatoric strain. Equation 4.33 is the deviatoric stress-strain relation. Note that:

\[
G = \mu
\]  

(4.34)

By combining Eqs. 4.32 and 4.33, the inverse stress-strain relation is easily derived as

\[
e_{ij} = e_{ij} + \frac{1}{3} \epsilon_{kk} \delta_{ij}
\]

\[
= \frac{1}{2\mu} s_{ij} + \frac{1}{3(3\lambda + 2\mu)} \sigma_{kk} \delta_{ij}
\]

\[
= \frac{1}{2\mu} (\sigma_{ij} - \frac{1}{3} \sigma_{kk} \delta_{ij}) + \frac{1}{3(3\lambda + 2\mu)} \sigma_{kk} \delta_{ij}
\]

\[
= \frac{1}{2\mu} \sigma_{ij} + \left[ \frac{6\mu - 9\lambda - 6\mu}{18(3\lambda + 2\mu)\mu} \right] \sigma_{kk} \delta_{ij}
\]

\[
e_{ij} = \frac{1}{2\mu} \sigma_{ij} - \frac{\lambda}{2\mu(3\lambda + 2\mu)} \sigma_{kk} \delta_{ij}
\]

(4.35a)

Or

\[
e_{ij} = \left[ -\frac{\lambda}{2\mu(3\lambda + 2\mu)} \delta_{y} \delta_{kl} + \frac{1}{\mu} (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}) \right] \sigma_{kl}
\]

(4.35b)

\[
e_{ij} = S_{ijk} \sigma_{kl} ; \quad S_{ijk} = -\frac{\lambda}{2\mu(3\lambda + 2\mu)} \delta_{y} \delta_{kl} + \frac{1}{\mu} (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk})
\]

(4.35c)

where \( S_{ijk} \sigma_{kl} \) is the compliance tensor.

In a matrix-vector form:

\[
\begin{bmatrix}
\epsilon_{11} \\
\epsilon_{22} \\
\epsilon_{33} \\
\epsilon_{12} \\
\epsilon_{23} \\
\epsilon_{13}
\end{bmatrix} =
\begin{bmatrix}
\alpha & \beta & \beta & 0 & 0 & 0 \\
\beta & \alpha & \beta & 0 & 0 & 0 \\
\beta & \beta & \alpha & 0 & 0 & 0 \\
0 & 0 & 0 & \gamma & 0 & 0 \\
0 & 0 & 0 & 0 & \gamma & 0 \\
0 & 0 & 0 & 0 & 0 & \gamma
\end{bmatrix}
\begin{bmatrix}
\sigma_{11} \\
\sigma_{22} \\
\sigma_{33} \\
\sigma_{12} \\
\sigma_{23} \\
\sigma_{13}
\end{bmatrix}
\]

(4.35a)

where

\[
\alpha = \frac{\lambda + \mu}{\mu(3\lambda + 2\mu)} ; \quad \beta = -\frac{\lambda}{2\mu(3\lambda + 2\mu)} ; \quad \gamma = \frac{1}{2\mu}
\]

(4.35b)

The Young’s modulus, \( E \), and the Poisson’s ratio \( \nu \), are related to the bulk modulus and shear modulus as
\[
K = \frac{E}{3(1-2\nu)}; \quad G = \frac{E}{3(1+\nu)} \quad (4.36)
\]

### 4.4. Limits on Elastic Parameter Values

We will consider only isotropic materials in this section. The bounds for the values of elastic parameters may be established from the principles of thermodynamics or from physical observations; we use the latter approach here. For finite (i.e., not infinite) strains, the stress is finite, and for finite stress, the strain is finite. This leads to:

\[
E, G, K \neq \infty; \quad E, G, K \neq 0 \quad (4.37)
\]

When a material is subjected to (1) a compressive linear stress, the material experiences a compressive linear strain; (b) a compressive hydrostatic stress, the material experiences a compressive volumetric strain, and (c) a shear stress, the material experiences a shear strain of the same sign. These observations yield

\[
E, G, K > 0 \quad (4.38)
\]

Combining Eqs. 4.37 and 4.38:

\[
0 < E, G, K < \infty \quad (4.39)
\]

The combination of the condition 4.39 and the relations in 4.36, one obtains the following restriction for the Poisson’s ratio:

\[-1 < \nu < \frac{1}{2}\]

#### Table 4.1 Examples of the Values of Elastic Parameters

<table>
<thead>
<tr>
<th>Material</th>
<th>(E) (GPa)</th>
<th>(\nu)</th>
</tr>
</thead>
<tbody>
<tr>
<td>rubber</td>
<td>0.00175-0.00245</td>
<td>(\sim 0.5)</td>
</tr>
<tr>
<td>Aluminum (cast)</td>
<td>87.5</td>
<td>0.34</td>
</tr>
<tr>
<td>Brass</td>
<td>84</td>
<td>0.33</td>
</tr>
<tr>
<td>mild steel</td>
<td>210</td>
<td>0.28</td>
</tr>
<tr>
<td>Glass</td>
<td>56-74</td>
<td>0.23</td>
</tr>
<tr>
<td>Spruce (15% moisture)</td>
<td>9.1</td>
<td>0.08</td>
</tr>
</tbody>
</table>

Some examples of the values of \(E\) and \(\nu\) are presented in Table 4.1 (Eringen, 1967) for a few materials. No material has a Poisson’s ratio of exactly 0.5, but rubber has a value close to 0.5. The Poisson’s ratio of coke is approximately 0. Almost all materials have a positive Poisson’s ratio value, except some recently developed materials.
4.5. Loading in Non-Principal Material Directions

The simpler relations derived in Section 4.2 for anisotropic materials are valid only when the loading axes coincide with the principal material directions. For example, the orthotropic relations (Eqs. 4.17, 4.18, 4.20 and 4.21) are valid only when the loading directions coincide with the axes of the $x_1 - x_2 - x_3$ coordinate system shown in Fig. 3. Otherwise, a coordinate transformation is needed to obtain the relevant relation. Let:

$$\sigma_{ij} = C_{ijkl} \varepsilon_{kl}$$ in the principal material directions, and
$$\sigma'_{ij} = C'_{ijkl} \varepsilon'_{kl}$$ in the loading directions.

Since $C_{ijkl}$ is a tensor (Eq. 4.3c), the following inverse relation holds:

$$C'_{ijpq} = C_{k'lm'a_ja_{j'}a_{pm}a_{qn}}$$

(4.40)

Equation 4.40 can be used to calculate the stiffness tensor from the elastic parameters defined in the principal material directions. The coefficients of the $6 \times 6$ matrix $\mathbf{C}$ can then be generated as before. The matrix will still be symmetric, but the number of non-zero components will be higher than 13.
Problems

Problem 4.1

Determine the form of $\mathbf{\bar{C}}$ (Eq. 4.10b) when there is one plane of symmetry with respect to the $x_i = 0$ plane, i.e., the behavior is identical when the loading directions are obtained by reflected about the $x_i = 0$ plane. Ans:

$$
\mathbf{\bar{C}} = 
\begin{bmatrix}
C_{1111} & C_{1122} & C_{1133} & 0 & C_{1123} & 0 \\
C_{1122} & C_{2222} & C_{2233} & 0 & C_{2223} & 0 \\
C_{1133} & C_{2233} & C_{3333} & 0 & C_{3323} & 0 \\
0 & 0 & 0 & C_{1212} & 0 & C_{1213} \\
C_{2311} & C_{2322} & C_{2333} & 0 & C_{2323} & 0 \\
0 & 0 & 0 & C_{1312} & 0 & C_{1313}
\end{bmatrix}
$$

Problem 4.2

Equation 4.17 was obtained for an orthotropic material by imposing symmetries about the $x_3 = 0$ and $x_2 = 0$ planes. Show that a further consideration of symmetry with respect to the $x_1 = 0$ plane leaves Eq. 4.17 unchanged.

Problem 4.3

The properties of an orthotropic material are: $E_1 = 200$ GPa, $E_2 = 150$ GPa, $E_3 = 100$ GPa, $\nu_{12} = 0.25$, $\nu_{23} = 0.30$, $\nu_{13} = 0.35$, $G_{12} = 100$ GPa, $G_{23} = 80$ GPa and $G_{13} = 60$ GPa. Determine the stresses corresponding to the following strains:

$$
\mathbf{\varepsilon} = 
\begin{bmatrix}
0.01 & 0.005 & 0 \\
0.005 & 0.02 & 0 \\
0 & 0 & 0.03
\end{bmatrix}
$$

What are the mean normal pressure and the deviatoric stress invariant (i.e., $J$, Chapter 2)? Ans:

$\bar{C}_{11} = 270.63$ GPa, $\bar{C}_{12} = 105.58$ GPa, $\bar{C}_{13} = 126.39$ GPa

$\bar{C}_{22} = 195.72$ GPa, $\bar{C}_{23} = 86.43$ GPa

$\bar{C}_{33} = 139.40$ GPa

$\overline{\sigma} = \{8.61, 7.56, 7.17, 0.6, 0, 0\}$

$p = 7.78$ GPa

$J = 0.956$ GPa.
Problem 4.4

Verify that with the transformation given by Eq. 4.23a, the stiffness matrix given by Eq. 4.22b reduces to the one given by Eq. 4.23b. Ans:
\[
\begin{align*}
\bar{C}_{11} &= C_{1111} = +C_{2222}, & \bar{C}_{12} &= C_{1122} = +C_{2211}, & \bar{C}_{13} &= C_{1133} = +C_{2233}, & \bar{C}_{14} &= C_{1112} = -C_{2221} \\
\bar{C}_{22} &= C_{2222} = +C_{1111}, & \bar{C}_{23} &= C_{2233} = +C_{1133}, & \bar{C}_{24} &= C_{2212} = -C_{1121}, & \bar{C}_{33} &= C_{3333} = +C_{1333} \\
\bar{C}_{34} &= C_{3312} = -C_{3231}, & \bar{C}_{44} &= C_{1212} = +C_{2121}, & \bar{C}_{55} &= C_{2323} = +C_{1313}, & \bar{C}_{56} &= C_{2313} = -C_{1323} \\
\bar{C}_{66} &= C_{1313} = +C_{2323}
\end{align*}
\]

These results, in conjunction with the symmetries such as \( C_{1323} = C_{2313} \), lead to the required result.

Problem 4.5

Assuming that the stiffness matrix \( \bar{C} \) of Problem 4.3 is given in the \( x_1 - x_2 - x_3 \) coordinate system, find the values of its coefficients in a rotated coordinate system \( x'_1 - x'_2 - x'_3 \) coordinate system obtained by rotating \( x_1 - x_2 - x_3 \) system about the \( x_2 \)-axis through an angle of \( 45^\circ \) clockwise. Ans:

\[
\bar{C} = \begin{bmatrix}
C_{1111} & C_{1112} & C_{1133} & 0 & 0 & 0 \\
C_{1122} & C_{1122} & C_{1233} & 0 & 0 & 0 \\
C_{1133} & C_{2222} & C_{2233} & 0 & 0 & 0 \\
0 & 0 & 0 & C_{1212} & 0 & 0 \\
0 & 0 & 0 & 0 & C_{2323} & 0 \\
0 & 0 & 0 & 0 & 0 & C_{1313}
\end{bmatrix} = \begin{bmatrix}
270.6 & 105.6 & 126.4 & 0 & 0 & 0 \\
105.6 & 195.7 & 86.4 & 0 & 0 & 0 \\
126.4 & 195.7 & 139.4 & 0 & 0 & 0 \\
0 & 0 & 0 & 100 & 0 & 0 \\
0 & 0 & 0 & 0 & 80 & 0 \\
0 & 0 & 0 & 0 & 0 & 60
\end{bmatrix}
\]

\[
\bar{C}' = \begin{bmatrix}
C'_{1111} & C'_{1112} & C'_{1133} & C'_{1112} & C'_{1123} & C'_{1113} \\
C'_{1211} & C'_{1222} & C'_{1233} & C'_{1212} & C'_{1223} & C'_{1213} \\
C'_{3111} & C'_{3122} & C'_{3233} & C'_{3112} & C'_{3123} & C'_{3113} \\
0 & 0 & 0 & C'_{1212} & 0 & 0 \\
0 & 0 & 0 & 0 & C'_{2323} & 0 \\
0 & 0 & 0 & 0 & 0 & C'_{1313} \\
\end{bmatrix} = \begin{bmatrix}
180.7 & 96.0 & 150.7 & 0 & 0 & 17.8 \\
96.0 & 195.7 & 96.0 & 0 & 0 & 9.6 \\
150.7 & 96.0 & 180.7 & 0 & 0 & 47.8 \\
0 & 0 & 0 & 30 & 0 & 0 \\
0 & 0 & 0 & 0 & 40 & 0 \\
17.8 & 9.6 & 47.8 & 0 & 0 & 54.3
\end{bmatrix}
\]

Using the strain vector:
\( \bar{\varepsilon}' = \{0.01\ 0.02\ 0.03\ 0\ 0\} \)
the corresponding stress vector is
\( \bar{\sigma}' = \{8.25\ 7.75\ 8.85\ 0\ 0\ 1.8\} \)

Note the development of a shear stress.
The following algorithm may be used for computer implementation:

1. Define and initialize the following arrays
   a. Stress array $S(6)$
   b. Strain array $E(6)$
   c. Stiffness tensor array (in $x_1 - x_2 - x_3$ system) $C(3,3,3,3)$
   d. Transformation matrix array $A(3,3)$
   e. An integer array $II(6)=(11,22,33,12,23,13)$

2. Read in or type in the values of $E(6)$, $C(3,3,3,3)$ and $A(3,3)$

3. Calculate $\bar{C}'$ using the following algorithm
   a. DO 1000 $M=1,6$
   b. $I=II(M)/10$
   c. $J=\mod(II(M),10)$
   d. DO 1000 $N=1,6$
   e. $K=II(N)/10$
   f. $L=\mod(II(N),10)$
   g. $\text{CBAR}(M,N)=0.0$
   h. DO 1000 $NP=1,3$
   i. DO 1000 $NQ=1,3$
   j. DO 1000 $NR=1,3$
   k. DO 1000 $NS=1,3$
   l. $\text{CBAR}(M,N)=\text{CBAR}(M,N)+C(NP,NQ,\text{NR},\text{NS}) \times A(NP,I) \times A(NQ,J) \times A(\text{NR} \times K) \times A(\text{NS},L)$
   m. 1000 CONTINUE

4. Calculate the stress vector in the $x_1' - x_2' - x_3'$ system as
   a. DO 2000 $I=1,6$
   b. $S(I)=0.0$
   c. DO 2000 $K=1,6$
   d. $S(I)=S(I)+\text{CBAR}(I,K) \times E(K)$
   e. 2000 CONTINUE

5. Print results