3.1. Introduction

The finite element method is a numerical method for obtaining approximate solutions for boundary value problems that are defined in terms of a set of governing mathematical equations, boundary (and/or initial conditions), constitutive laws and other secondary relations such as the strain-displacement relations. In general, a number of different principles (e.g., virtual work principle, minimum potential energy principle, variational principle, etc.) are used as the basis for developing the finite element equations for a given boundary value problem. Regardless of the method used, the following aspects remain the same (referring to Fig. 1):

The domain is divided into a number of elements, interconnected at a set of nodes.

The primary unknowns at a given point within the element is expressed in terms of their nodal values with the aid of piece-wise shape functions, as

\[ \mathbf{u} = \mathbf{N}\hat{\mathbf{u}} \]  

(1a)

where \( \mathbf{u} \) is the vector containing the primary unknowns at any point within the element, \( \hat{\mathbf{u}} \) is a vector containing the nodal unknowns surrounding that element and \( \mathbf{N} \) is a matrix containing the shape functions. The size of \( \mathbf{N} \) depends on the problem, and will be discussed in the context of specific problems. The secondary unknowns (e.g., strain \( \varepsilon \) in solid mechanics problems) are then derived from Eq. 1a and expressed in terms of \( \hat{\mathbf{u}} \) as well. For example, in solid mechanics problem, the strain vector is expressed in terms of \( \hat{\mathbf{u}} \) as

\[ \varepsilon = \mathbf{B}\hat{\mathbf{u}} \]  

(1b)

Using the chosen principle as the basis, the nodal unknowns \( \hat{\mathbf{u}} \) are determined and the secondary variable are then evaluated using relations such as Eq. 1b.
With this cursory introduction to the finite element method, we now illustrate some of the principles that are suitable for finite element modeling of solid mechanics problems.

![Diagram](image)

**Fig. 2. Notations for Variables of Elasticity Problems**

### 3.2. Bases for Finite Element Equations

#### 3.2.1. Virtual Work Method

A principle that is widely used in solving solid mechanics problems is the virtual work principle. The virtual work principle is a variation of the equilibrium equations and boundary conditions (Eq. 23a-23e, Chapter 2) involved in the definition of boundary value problem in solid mechanics, and the connection is described in Appendix 2. The notations used for the variables are presented in Fig. 2. Let us impose on the domain a virtual displacement vector \( \hat{u} \). Let \( \hat{\varepsilon} \) be the strain (virtual strain) vector caused by the imposition of the virtual displacement \( \hat{u} \). \( \hat{u} \) must be a “pertinent kinematically admissible” field; i.e., \( \hat{u} \) must satisfy the following conditions:

- \( \hat{u} = 0 \) on \( \Gamma_u \) and
- Smooth enough for \( \hat{\varepsilon} \) to be derived from it.

The virtual work principle states

\[
\text{The internal virtual work} - \text{External virtual work} = 0
\]

\[
\int_\Omega \hat{\varepsilon}^T \sigma dv - \int_\Omega \hat{\varepsilon}^T b dv - \int_{\Gamma_b} \hat{\varepsilon}^T t ds = 0 \tag{2}
\]

Since \( \hat{u} = 0 \) over \( \Gamma_u \), the boundary term includes only the integral over \( \Gamma_b \). \( b \) is the body force vector. The first term represents the internal virtual work and the remaining two terms represent the virtual external work.

Eq. 2 provides us with a basis for determining the unknowns of the problem. Taking the displacements as the primary unknown variables, we approximate the displacement field
by a set of unknowns, express all other secondary variables of the problem (\(\sigma\) and \(\varepsilon\) in the present case) in terms of the primary unknowns, and use Eq. 2 to determine the primary unknowns.

In the finite element analysis, the integrals in Eq. 2 are broken up into integrals over each finite element \(\Omega_i\) as

\[
\sum_{i=1}^{N_E} \int_{\Omega_i} \delta \varepsilon^T \sigma dv - \sum_{i=1}^{N_E} \int_{\Gamma_i} \delta \mathbf{u}^T \mathbf{b} dv - \int_{\Gamma_e} \delta \mathbf{t}^T \mathbf{t} ds = 0
\]

(3)

where \(N_E\) is the number of finite elements in the domain. Consider a typical term in the first integral, associated with a typical element \(\Omega_i\). From Eqs. 1a and 1b, denoting the vector containing the nodal unknowns surrounding element \(i\) as \(\hat{\mathbf{u}}_i\), one has:

\[
\mathbf{u} = N \hat{\mathbf{u}}_i \tag{4a}
\]

\[
\varepsilon = B \hat{\mathbf{u}}_i \tag{4b}
\]

and

\[
\delta \mathbf{u} = N \delta \hat{\mathbf{u}}_i \tag{4c}
\]

\[
\delta \varepsilon = B \delta \hat{\mathbf{u}}_i \tag{4d}
\]

In Eq. 4a, an assumption is made concerning the variation of the displacement within a given element. The “order” of the shape function is chosen so that the integrals in Eq. 2 will not become infinite. Eq. 2 involves stresses, which, in linear elastic materials, are linearly related to strains, which in turn are derived from the first derivatives of the displacements (recall the strain-displacements presented in Chapter 2). Thus, the first derivative of displacement must not become infinite, but the \(2^{nd}\) and higher derivatives can be infinite. Such functions are called \(C_0\) continuous functions (i.e., the function is continuous, its first derivative is finite but not necessarily continuous, as shown in Fig. 2b. This can be generalized to state that if the highest order of derivatives in the integral is \(n\), the shape function must be at least \(C_{n-1}\) continuous.

For an elastic material

\[
\sigma = D \varepsilon \tag{4e}
\]

The first integral then becomes:

\[
I_1^i = \int_{\Omega_i} \delta \varepsilon^T \sigma dv = \int_{\Omega_i} \delta \hat{\mathbf{u}}_i^T \mathbf{B}^T \mathbf{u} dv = \int_{\Omega_i} \mathbf{B}^T \mathbf{D} \mathbf{B} \mathbf{u} dv \hat{\mathbf{u}}_i = \mathbf{B}^T \mathbf{D} \mathbf{B} \hat{\mathbf{u}}_i \tag{5a}
\]

where

\[
k_i = \int_{\Omega_i} \mathbf{B}^T \mathbf{D} \mathbf{B} dv \tag{5b}
\]

is referred to as the element stiffness matrix.

A typical term in the second integral takes the form

\[
I_2^i = \int_{\Omega_i} \delta \mathbf{u}_i^T \mathbf{b} dv = \int_{\Omega_i} \mathbf{N}^T \mathbf{b} dv = \int_{\Omega_i} \mathbf{N}^T \mathbf{f}_i^b
\]

(6a)

where
\( f_i^b = \int_{\Omega} N^T b dv \) \hspace{1cm} (6b)

is referred to as the element force vector due to body force. Even the 3\textsuperscript{rd} term in Eq. 3 can be split up into elemental components. Each elemental term leads to element force vector due to boundary traction \( f_i' \) as

\[
I_i^3 = \int_{\Gamma_i} \delta u_i^T t ds = \delta u_i^T \int_{\Gamma_i} N^T t ds = \delta u_i^T f_i'
\]

where

\[
f_i' = \int_{\Gamma_i} N^T t ds \hspace{1cm} (7b)
\]

Note that the boundary integral in Eq. 7b is performed only over the boundary that surrounds element \( i \). Eq. 3 now becomes:

\[
\sum_{i=1}^{N_E} I_i^1 - \sum_{i=1}^{N_E} I_i^2 - \sum_{i=1}^{N_E} I_i^3 = 0 \hspace{1cm} (8a)
\]

\[
\sum_{i=1}^{N_E} \delta u_i^T k_i \hat{u}_i - \sum_{i=1}^{N_E} \delta u_i^T f_i^b - \sum_{i=1}^{N_E} \delta u_i^T f_i' = 0 \hspace{1cm} (8b)
\]

Fig. 3. Definition of a One-Dimensional Rod Problem

There is a systematic way of performing the summations involved in Eq. 8b, which we will discuss in detail later. Very briefly, a global primary unknown nodal displacement \( \hat{u} \) is defined. Consistent with the definition of \( \hat{u} \), a global stiffness matrix \( K \) and a global load vector \( P \) are defined. \( K \) and \( P \) are formed by assembling \( k_i \), \( f_i^b \) and \( f_i' \) for \( i = 1, N_E \). Eq. 8b then takes the form:

\[
\delta \hat{u}^T (K \hat{u} - P) = 0 \hspace{1cm} (9)
\]
If Eq. 9 is to be true for any arbitrary and nontrivial (nonzero) virtual displacement $\delta u$, then

$$K\delta u = P$$

(10)

Thus, a set of linear simultaneous equations are obtained. At this stage, the displacement boundary conditions are imposed and a suitable solution method is used to solve for $\delta u$.

It can be shown that in the displacement method, the structure is modeled to be stiffer than the actual structure; i.e., the calculated displacements are smaller than the actual displacements. As the number of elements are increased, the solution approaches exact solution for $N_E \geq N_{E_k}^\ast$, where $N_{E_k}^\ast$ varies with the actual nonlinearity of the spatial variation of displacements, and the order of shape functions employed in the development of the element stiffness matrices. For instance, in the problem of a axially loaded one-dimensional rod subjected to a point load at one end and supported at the other end, the displacement varies linearly with $x$ and thus one element based on linear shape function is adequate to model it exactly, i.e., $N_{E_k}^\ast = 1$.

**Example 3.1**

**Question:**
By following the virtual work-based procedure outlined in Section 3.2, solve for the displacements at points B and C for the axially-loaded, one-dimensional rod problem shown in Fig. 3.

**Answer:**
We begin by defining the matrices $N$ and $B$ for a typical element $i$. As the problem is one-dimensional, we represent the displacement vector at any point $x$ by a scalar $u(x)$. To develop the element-based equations, we define a local coordinate system $t$ as shown in Fig. 3b. Note that the positive direction of $t$ is taken from local node 1 to local node 2. The next step is to choose a shape function of an acceptable order. The shape function must be a polynomial of at least order 0. Here we choose a linear function (order 1), as shown in Fig. 3b. The displacement unknown is expressed as

$$u = N_1\delta u_1 + N_2\delta u_2 = N\delta u$$

(11)

where $N_1 = 1 - t/\ell$ and $N_2 = t/\ell$, and $N = \begin{bmatrix} N_1 & N_2 \end{bmatrix}$ is a $1 \times 2$ shape matrix (the $N$-matrix). Note the special characteristics of the shape functions: At local node 1, $N_1 = 1$ and $N_2 = 0$ and at local node 2, $N_1 = 0$ and $N_2 = 1$, resulting in $u = \delta u_1$ at node 1, and $u = \delta u_2$ at node 2.

From the strain-displacement relation

$$\varepsilon = \frac{du}{dx} = \frac{du}{dt} = B_1\delta u_1 + B_2\delta u_2 = B\delta u$$

(12)

where $B_1 = -1/\ell$ and $B_2 = 1/\ell$, and $B = \begin{bmatrix} B_1 & B_2 \end{bmatrix}$ is a $1 \times 2$ matrix (the $B$-matrix).

The relevant constitutive equation is

$$\sigma = E\varepsilon = D\varepsilon$$

(13)

where $E$ is the Young’s modulus, and $D = \begin{bmatrix} E \end{bmatrix}$ is the $1 \times 1$ material stiffness matrix.

From Eq. 5b:
where \( k = \frac{Ea}{L} \).

The reader may easily verify that in this one-dimensional problem, the surface traction may either be treated as a surface traction or as a body force; either way, one ends up with the same final results. This is, however, not true in general. We will treat the load as surface traction.

Noting that and expressing \( f_x \) as a function of \( t \) as

\[
f_x = f_i = \left(1 - \frac{t}{\ell}\right)f_i + \left(\frac{t}{\ell}\right)f_2 = N_1f_i + N_2f_2 \tag{15a}
\]

where

\[
f_i = f_0(1 - x_i/L) \quad \text{and} \quad f_2 = f_0(1 - x_2/L), \tag{15b}
\]

with \( x_i \) and \( x_2 \) representing the global \( x \)-coordinates of nodes 1 and 2 of the typical element \( i \), and \( f_i \) and \( f_2 \) are the values of \( f \) at nodes 1 and 2. Note that the particular form of Eq. 15a in terms of \( N_1 \) and \( N_2 \) is unique to the specific loading and not a general result. Now, from Eq. 7b:

\[
f_i' = \int_0^\ell \begin{bmatrix} N_1 \\ N_2 \end{bmatrix} f_x dt = \int_0^\ell \begin{bmatrix} N_1 \\ N_2 \end{bmatrix} \left[ N_1 f_i + N_2 f_2 \right] dt = \frac{\ell}{6} \begin{bmatrix} 2f_i + f_2 \end{bmatrix} \tag{16}
\]

Eqs. 14 and 16 are generalized expressions for the element stiffness matrix and the load vector due to surface traction, and can be used to evaluate the respective quantities for any given element.

Let us divide the rod into 3 elements of equal length as shown in Fig. 3d. There is no theoretical restriction for the choice of local directions. Just for generality and to demonstrate the manner in which global matrices are assembled from element matrices for general cases, we take the positive local directions of elements 1, 2 and 3 to be from global nodes 1 to 2, global nodes 4 to 3, and global nodes 2 to 3 respectively. The element stiffness and load vectors for each of the 3 elements are:

\[
k_i = k_2 = k_3 = \begin{bmatrix} k & -k \\ -k & k \end{bmatrix} \text{ where } k = \frac{3Ea}{L} \tag{17a}
\]

\[
f_1' = \begin{bmatrix} 8 \\ 7 \end{bmatrix}; \quad f_2' = \begin{bmatrix} 1 \\ 2 \end{bmatrix}; \quad f_3' = \begin{bmatrix} 5 \\ 4 \end{bmatrix} \text{ where } \bar{f} = \frac{Lf_0}{18} \tag{17b}
\]

Now the element matrices are assembled into the global matrices. To achieve this, correspondence between the two nodes surrounding each element and the global nodes are found. In this example, nodes 1 and 2 of element 1 go with global nodes 1 and 2 respectively, those of element 2 go with global nodes 2 and 3 respectively, and those of element 3 go with global nodes 3 and 4 respectively. At this point, let us define a global displacement vector as

\[
\mathbf{\hat{u}} = \begin{bmatrix} \hat{u}_1, \hat{u}_2, \hat{u}_3, \hat{u}_4 \end{bmatrix}
\tag{18}
\]

Denoting the stiffness matrix of element 1 by:

\[
k_1 = \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix}\text{ etc.,} \tag{19}
\]

the first summation in Eq. 8b can be written as
The 4×4 matrix in the middle of Eq. 20c is the global stiffness matrix \( K \).

**Remark: Assembling Rule**

The coefficients of a given element matrix may be directly placed into the global stiffness matrix by following the rule illustrated on Fig. 4. Here, the local nodes 1 and 2 of element \( i \) coincide with global nodes \( \ell \) and \( k \) respectively. Then, the 1\(^{st}\) and 2\(^{nd}\) rows of \( k \) get placed on \( \ell \)\(^{th}\) and \( k \)\(^{th}\) rows of the global matrix, and the 1\(^{st}\) and 2\(^{nd}\) columns of \( k \) get placed on \( \ell \)\(^{th}\) and \( k \)\(^{th}\) columns of the global matrix. The user may convince himself/herself that positioning of the coefficients of \( k \) in the global matrix comply with this rule.

\[
\begin{bmatrix}
  k & -k & 0 & 0 \\
  -k & 2k & -k & 0 \\
  0 & -k & 2k & -k \\
  0 & 0 & -k & k
\end{bmatrix}
\]

[\( \ddot{\mathbf{u}}_1, \ddot{\mathbf{u}}_2, \ddot{\mathbf{u}}_3, \ddot{\mathbf{u}}_4 \)]

\[=\begin{bmatrix}
  k & k_1^{11} & 0 & 0 \\
  k & k_1^{12} & k_2^{12} & 0 \\
  0 & k_2^{21} & k_2^{22} & k_2^{23} \\
  0 & 0 & k_1^{12} & k_1^{11}
\end{bmatrix}
\begin{bmatrix}
  \ddot{u}_1 \\
  \ddot{u}_2 \\
  \ddot{u}_3 \\
  \ddot{u}_4
\end{bmatrix}
\]

\[=\begin{bmatrix}
  \ddot{u}_1 \\
  \ddot{u}_2 \\
  \ddot{u}_3 \\
  \ddot{u}_4
\end{bmatrix}
\]

The global load vector is similarly assembled from the element load vectors. Denoting the load vector of element 1 by (dropping the superscript \( t \) to avoid confusion):

\[ f_1 = \begin{bmatrix} f_1^1 \\ f_1^2 \end{bmatrix}, \text{ etc,} \]

\[ (21) \]
the 3\textsuperscript{rd} term in Eq. 8b becomes:

\[
\begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & f_2' & f_3' & 0 \\
0 & 0 & f_2' & 0 \\
0 & f_2' & 0 & 0
\end{bmatrix}
\]

\text{(22a)}

\[
\begin{bmatrix}
f_1' \\
\frac{f_1' + f_3'}{f_2'} \\
\frac{f_2' + f_3'}{f_2'} \\
\frac{f_1'}{f_2'}
\end{bmatrix}
\text{(22b)}

\[
\begin{bmatrix}
8\hat{f} \\
12\hat{f} \\
6\hat{f} \\
1\hat{f}
\end{bmatrix}
\text{(22c)}

The column vector in Eq. 22c is the global load vector. Thus, the global equation becomes:

\[
\begin{bmatrix}
k & -k & 0 & 0 \\
-k & 2k & -k & 0 \\
0 & -k & 2k & -k \\
0 & 0 & -k & k
\end{bmatrix}
\begin{bmatrix}
\hat{u}_1 \\
\hat{u}_2 \\
\hat{u}_3 \\
\hat{u}_4
\end{bmatrix}
= \begin{bmatrix}
8\hat{f} \\
12\hat{f} \\
6\hat{f} \\
1\hat{f}
\end{bmatrix}
\text{(23)}

The displacement boundary condition that

\[
\hat{u}_1 = 0
\text{(24a)}

is imposed at this stage. This may be achieved by simply dropping the 1\textsuperscript{st} column and 1\textsuperscript{st} row from Eq. 23. One then gets the following equation:

\[
\begin{bmatrix}
2k & -k & 0 \\
-k & 2k & -k \\
0 & -k & k
\end{bmatrix}
\begin{bmatrix}
\hat{u}_1 \\
\hat{u}_2 \\
\hat{u}_3
\end{bmatrix}
= \begin{bmatrix}
12\hat{f} \\
6\hat{f} \\
1\hat{f}
\end{bmatrix}
\text{(24b)}

\[
\begin{bmatrix}
2 & -1 & 0 \\
-1 & 2 & -1 \\
0 & -1 & 1
\end{bmatrix}
\begin{bmatrix}
\hat{u}_1 \\
\hat{u}_2 \\
\hat{u}_3
\end{bmatrix}
= \frac{\hat{f}}{k} \begin{bmatrix}
12 \\
6 \\
1
\end{bmatrix}
\text{(24c)}

which can be easily solved for the unknown displacements.
Exercise 3.1

Use the following quadratic shape functions
\[ \hat{u} = N_1 \hat{u}_1 + N_2 \hat{u}_2 + N_3 \hat{u}_3 \]  
(25a)

where
\[
N_1 = -\frac{t}{\ell} \left( 1 - \frac{2t}{\ell} \right); \quad N_2 = \frac{t}{\ell} \left( 1 + \frac{2t}{\ell} \right); \quad N_3 = 1 - \frac{4t^2}{\ell^2}
\]  
(25b)

with three nodes 1, 2 and 3, and the local axis \( t \) taken as shown in Fig. 5. For the problem shown in Fig. 3, derive a general equation for the element stiffness matrix and traction load vector. Divide the rod into two elements, one from point A to point B and another from point B to point C, and form the global stiffness and load vectors. Impose the displacement boundary condition and solve for the displacement at points B and C. Compare your results with those obtained in Example 3.1 and comment on the results.
3.3. Isoparametric Formulation of Two-Dimensional Elasticity Problems

3.3.1. Introduction

By following a procedure very similar to that involved in the development of the one-dimensional rod element in example 3.1, elements of various shapes such as triangular, rectangular, cubical, etc., can easily be developed. Triangular elements are fairly widely employed in the finite element analyses of solid mechanics problems. The reader is referred to standard texts on finite element methods for further details on these elements. An element type that is even more widely used is the so-called *isoparametric element*. This element allows domains with curved boundaries to be discretized in a more natural manner than elements that are restricted to have straight edges (e.g., triangular). The element employs a technique of transformation in dealing with curved boundaries and irregular shapes of element volumes (or areas). In isoparametric formulation, standard Cartesian coordinates are transformed into curvilinear coordinates (also called natural or intrinsic coordinates), as shown in Fig. 7, where \( \xi \) and \( \eta \) are curvilinear coordinates. They are normalized so that their values range between -1 and 1 as

\[
-1 \leq \xi \leq 1 \quad (80a) \\
-1 \leq \eta \leq 1 \quad (80b)
\]

Several types of isoparametric elements are available for modeling one-, two- and three-dimensional problems (e.g., 4-noded, bilinear, quadrilateral elements for modeling 2-dimensional problems, Fig. 8a, 20-noded, incomplete quadratic, hexahedral elements for modeling three-dimensional problems, Fig. 8b, etc.) A comprehensive list of some of the standard elements, along with pertinent equations, is summarized in Appendix 2. Equations associated with these different types of elements are similar. Once the fundamentals are understood for one type of element, it is easy to follow the details of the formulation for another. We will describe the details for an 8-noded, quadratic, quadrilateral, two-dimensional element, shown in Fig. 7, abbreviated as Q8 (8-noded, quadrilateral). Many of the details are the same regardless of whether the number of unknowns at each node is 1 (e.g., water flow problem, where the unknown is \( h \)), 2 (e.g., two-dimensional elasticity problems treated here, where the unknowns are the \( x \)- and \( y \)-components of the displacements \( u \) and \( v \)), 3 (e.g., two displacements and pore water pressure), etc. However, to avoid confusion, we will develop the equations for the two-dimensional elasticity problems.

To keep confusions to a minimum, we will develop the equations for a plane strain elasticity problem, but point out the modifications needed to extend the formulation for plane stress and axisymmetric problems.

Note that Q8 not only has nodes at the corners, but also has mid-nodes as shown in Fig. 7. Q8 does not have nodes within the element (Note that there are elements that have intra-element nodes). The values of \((\xi, \eta)\) at the 8 nodes are listed in Fig. 7; i.e.,
\( (\xi_i, \eta_i) \) for \( i=1,8 \). The \( x \)- and \( y \)-coordinates of any point within the element is expressed in terms of the values of the nodes as

\[
x = \sum_{i=1}^{8} N_i x_i \tag{81a}
\]

\[
y = \sum_{i=1}^{8} N_i y_i \tag{81b}
\]

where \((x_i, y_i)\) are nodal coordinates with respect to the global \( x \)- and \( y \)-coordinate axes, and \( N_i \) are isoparametric shape functions given by

\[
N_i = \frac{1}{4} (1 + \xi_i \xi)(1 + \eta_i \eta)(\xi_i \xi + \eta_i \eta - 1) \quad \text{for } i = 1,2,3,4 \tag{82a}
\]

\[
N_i = \frac{1}{2} (1 - \xi^2)(1 + \eta_i \eta) \quad \text{for } i = 5,7 \quad (\text{i.e., for } \xi = 0) \tag{82b}
\]

\[
N_i = \frac{1}{2} (1 - \eta^2)(1 + \xi_i \xi) \quad \text{for } i = 6,8 \quad (\text{i.e., for } \eta = 0) \tag{82c}
\]

In isoparametric formulation, one set of shape functions are used for the interpolation of coordinates (Eqs. 81a and 81b) and the displacements. There are two degrees of freedom (d.o.f.) at every point \((u,v)\), and thus at every nodes, \((\hat{u}_i, \hat{v}_i)\). The interpolation relations for the displacements are:

\[
u = \sum_{i=1}^{8} N_i \hat{v}_i \tag{83b}
\]

In a matrix form:

\[
\begin{bmatrix}
\hat{u}_1 \\
\hat{v}_1 \\
\hat{u}_2 \\
\hat{v}_2 \\
\vdots \\
\hat{u}_8 \\
\hat{v}_8
\end{bmatrix} = 
\begin{bmatrix}
N_1 & 0 & N_2 & 0 & \cdots & 0 & N_8 & 0 \\
0 & N_1 & 0 & N_2 & \cdots & 0 & 0 & N_8 \\
0 & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0
\end{bmatrix} 
\begin{bmatrix}
\hat{u}_1 \\
\hat{v}_1 \\
\hat{u}_2 \\
\hat{v}_2 \\
\vdots \\
\hat{u}_8 \\
\hat{v}_8
\end{bmatrix} \tag{84a}
\]

In a compact format:

\[
u = N \hat{u} \tag{84b}
\]

It is clear that the shape functions are continuous within an element. It is easily verified that the displacement field is continuous along inter-element boundaries as well. For instance, consider the boundary containing nodes 2, 6 and 3, where \( \xi = 1 \), and \( -1 \leq \eta \leq 1 \). Considering for instance, the horizontal displacement \( u \), it is easily verified that
Thus, \( u \) varies quadratically with \( \eta \) on this boundary and is uniquely defined in terms of the values of \((\hat{u}_2, \hat{u}_3, \hat{u}_6)\). Thus, as long as the adjacent elements share the same set of \((\hat{u}_2, \hat{u}_3, \hat{u}_6)\), then the displacement field is continuous at inter-element boundaries. That each element surrounding a given global node shares the same displacement unknown value is a fundamental feature of the finite element method. (However, sometimes there are reasons for employing elements which violate the inter-element continuity requirement. Such elements are known as nonconforming elements.)

Defining

\[
F_i = \frac{\partial N_i}{\partial x} \quad \text{(85a)}
\]

and

\[
G_i = \frac{\partial N_i}{\partial y} \quad \text{(85b)}
\]

The strain vector can be written as:

\[
\begin{bmatrix}
\epsilon_{xx} \\
\epsilon_{yy} \\
\gamma_{xy}
\end{bmatrix}_{3\times1} = \begin{bmatrix}
\frac{\partial u}{\partial x} \\
\frac{\partial v}{\partial y} + \frac{\partial v}{\partial x}
\end{bmatrix} = \begin{bmatrix}
F_1 & 0 & F_2 & 0 & \ldots & F_8 & 0 \\
0 & G_1 & 0 & G_2 & \ldots & 0 & G_8 \\
G_1 & F_1 & G_2 & F_2 & \ldots & G_8 & F_8
\end{bmatrix} \begin{bmatrix}
\hat{u}_1 \\
\hat{v}_1 \\
\hat{u}_2 \\
\hat{v}_2 \\
\vdots \\
\hat{u}_8 \\
\hat{v}_8
\end{bmatrix}
\]

\[
\text{(85c)}
\]

In a more compact format:

\[
\epsilon = \mathbf{B} \hat{\mathbf{u}}
\]

\[
\text{(85d)}
\]

Equations for \( F_i \) and \( G_i \) depend on (shown below) the derivatives of \( N_i \) w.r.t. \( \xi \) and \( \eta \), which are:

For \( i=1,2,3,4 \)

\[
\frac{\partial N_i}{\partial \xi} = \frac{\xi_i}{4}(1 + \eta \eta_i)(2 \xi_\xi \xi_i + \eta \eta_i) \quad \text{(86a)}
\]

\[
\frac{\partial N_i}{\partial \eta} = \frac{\eta_i}{4}(1 + \xi \xi_i)(\xi \xi \xi_i + 2 \eta \eta_i) \quad \text{(86b)}
\]

For \( i=5,7 \)

\[
\frac{\partial N_i}{\partial \xi} = -\xi(1 + \eta \eta_i) \quad \text{(86c)}
\]
\[
\frac{\partial N_i}{\partial \eta} = \frac{1}{2}(1-\xi^2)\eta_i, \quad \text{(86d)}
\]

For \(i=6,8\)
\[
\frac{\partial N_i}{\partial \xi} = \frac{1}{2}(1-\eta^2)\xi_i, \quad \text{(86e)}
\]
\[
\frac{\partial N_i}{\partial \eta} = -\eta(1+\xi\xi_i) \quad \text{(86f)}
\]

To see how the equations defer depending on whether it is plane strain or plane stress, let us for a moment go back to the potential energy expression given by Eq. 49b, and consider the first term:
\[
\pi_1 = \int_\Omega \sigma^t \varepsilon^c dV
\]
\[
= \int_\Omega \left[ (\sigma_{xx} \varepsilon_{xx} + \sigma_{yy} \varepsilon_{yy} + \sigma_{xy} \gamma_{xy}) + (\sigma_{zz} \varepsilon_{zz} + \sigma_{xz} \gamma_{xz} + \sigma_{yz} \gamma_{yz}) \right] dV
\]

For a plane strain problem, since \(\varepsilon_{zz} = \gamma_{zz} = 0\), the 2\textsuperscript{nd} groups of terms in Eq. 86 is zero and element stiffness matrix will be based on \(\varepsilon = [\varepsilon_{xx}, \varepsilon_{yy}, \gamma_{xy}]\) and \(\sigma = [\sigma_{xx}, \sigma_{yy}, \sigma_{xy}]\) and the 3\times3 \(D\) matrix given by Eqs. 19a and 19b of Chapter 2. For the plane stress problem, while \(\gamma_{xx} = \gamma_{yy} = 0\), \(\varepsilon_{zz} \neq 0\), which leads to a modification of the constitutive law (Exercise 2.2, Chapter 2). The element stiffness matrix is based on \(\varepsilon = [\varepsilon_{xx}, \varepsilon_{yy}, \gamma_{xy}]\) and \(\sigma = [\sigma_{xx}, \sigma_{yy}, \sigma_{xy}]\) and the 3\times3 \(D\) matrix given by Eqs. 19a and 19c of Chapter 2. In axisymmetric problems, all three normal components of the stresses and strains are nonzero, and thus the element stiffness matrix is based on \(\varepsilon = [\varepsilon_{rr}, \varepsilon_{\theta\theta}, \varepsilon_{rz}, \gamma_{rz}]\) and \(\sigma = [\sigma_{rr}, \sigma_{\theta\theta}, \sigma_{rz}, \sigma_{rz}]\) and the 4\times4 \(D\) matrix given by Eqs. 19a and 28 of Chapter 2.

Returning now to the plane strain problem, the element stiffness matrix is given by Eq. 5b, and the element load vectors due to body force and boundary traction are given by Eq. 6b and 7b respectively. The integrals need to be evaluated w.r.t. \(x\) and \(y\) coordinates, whereas the \(N\) and \(B\) matrices are in intrinsic coordinates, requiring transformation between the two systems. The relevant transformation is derived directly from Eq. 81a and 81b as:
\[
\begin{bmatrix}
\frac{dx}{dy} \\
\frac{dy}{dx}
\end{bmatrix}_{2\times1} = \begin{bmatrix}
\frac{\partial x}{\partial \xi} & \frac{\partial x}{\partial \eta} \\
\frac{\partial y}{\partial \xi} & \frac{\partial y}{\partial \eta}
\end{bmatrix}_{2\times2} \begin{bmatrix}
\frac{d\xi}{d\eta} \\
\frac{d\eta}{d\xi}
\end{bmatrix}_{2\times1} = \mathbf{J} \begin{bmatrix}
\frac{d\xi}{d\eta} \\
\frac{d\eta}{d\xi}
\end{bmatrix}_{2\times1}
\]
\[
\frac{dx dy}{d\xi d\eta} = |J| = \left| \begin{bmatrix}
\frac{d\xi}{d\eta} \\
\frac{d\eta}{d\xi}
\end{bmatrix}_{2\times1} \right|
\]
\[
\text{(87a)}
\]

where the \(2\times2\) matrix \(\mathbf{J}\) is called the Jacobian matrix. The area element needed in Eq. 5b is transformed as:
\[
d\xi d\eta = d\xi d\eta = |J| d\xi d\eta
\]
\[
\text{(87b)}
\]

That is, area integrals are transformed as:
\[
\int_{\Omega} \ldots d\xi d\eta = \int_{\Omega} \ldots |J| d\xi d\eta
\]
\[
\text{(87c)}
\]

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Note that
\[ \int dxdy = \int_{\Omega} |J| d\xi d\eta = \text{area of a plane element} \quad (87d) \]
and the 3D equivalent will give the volume and the 1D equivalent will give the length.

Note that in the case of axisymmetric approximation, the volume integrals involve:
\[ \int_{\Omega} (..) r dr d\theta dz = 2\pi \int_{\Omega} \left( \sum_{j=1}^{n} r_j \cdot N_j \right) |J| d\xi d\eta \quad (87e) \]

For the plane problems, by the chain rule of differentiation:
\[
\left[ \frac{\partial N_i}{\partial \xi}, \frac{\partial N_i}{\partial \eta} \right] = \left[ \frac{\partial x}{\partial \xi}, \frac{\partial y}{\partial \xi} \right] \left[ \frac{\partial N_i}{\partial \xi}, \frac{\partial N_i}{\partial \eta} \right] \quad (88a)
\]
where from Eqs. 81a and 81b:
\[
\frac{\partial x}{\partial \xi} = \sum_{i=1}^{8} \frac{\partial N_i}{\partial \xi} x_i \quad (88b)
\]
\[
\frac{\partial y}{\partial \xi} = \sum_{i=1}^{8} \frac{\partial N_i}{\partial \xi} y_i \quad (88c)
\]
\[
\frac{\partial x}{\partial \eta} = \sum_{i=1}^{8} \frac{\partial N_i}{\partial \eta} x_i \quad (88d)
\]
\[
\frac{\partial y}{\partial \eta} = \sum_{i=1}^{8} \frac{\partial N_i}{\partial \eta} y_i \quad (88e)
\]
and the derivatives of \( N_i \) w.r.t. the intrinsic coordinates are given in Eq. 86. Eq. 88a is inverted to find \( F_i \) and \( G_i \).

Integrations need to be performed in evaluating the element stiffness and load vectors. The expression for the element stiffness matrix is given in Eq. 5b, which for the plane strain problem of interest becomes:
\[
k_i = \int_{\Omega} B^T DB t dxdy \quad (89a)
\]
\[
= \int_{-1}^{1} \int_{-1}^{1} B^T DB t |J| d\xi d\eta \quad (89b)
\]
where \( t \) is the thickness (perpendicular to plane of loading) of the element, and is normally taken as unity. A typical term in Eq. 89b is in the form:
\[ I = \int_{-1}^{1} \int_{-1}^{1} F(\xi, \eta) \, d\xi \, d\eta \]  

(89c)

where \( F(\xi, \eta) \) is a scalar function of certain order in \( \xi \) and \( \eta \). Typically, both the numerator and the denominator of \( F(\xi, \eta) \) have polynomial terms, and as such, it is not possible to carry out the integration analytically. A numerical method needs to be employed. One of the most commonly used techniques is known as the Gauss method, and the reader is referred to standard texts (e.g., Kopal, 1955; Stroud and Secrest, 1966) or other finite element books (e.g., Cook, 1981) for theoretical details. In brief, the continuous integral in Eq. 89c is replaced by discrete sum as:

\[ I = \sum_{i=1}^{n} \sum_{j=1}^{n} w_i w_j F(\xi_i, \eta_j) \]  

(90)

where \( n \) is the order of integration, \( w_i \) and \( w_j \) are weights, and \( (\xi_i, \eta_j) \) are intrinsic coordinates at the sampling points (Gauss quadrature). In Gauss method, the sampling points and the weights are chosen so that for a given order of integration, the best results are obtained (i.e., error between the actual and numerical values of the integral is minimum). The sampling points for order 1, 2 and 3 integrations are shown in Figs. 9 and 10 for one- and two-dimensional elements respectively. The values of \( \xi_i \) and \( w_i \) up to order 5 are presented in Table 1.

Fig. 9. Sampling Points for Gauss Integration of One-Dimensional Elements

Fig. 10. Sampling Points for Gauss Integration of Two-Dimensional Elements
Table 1: Sampling Points and Weights for Gauss Quadrature

<table>
<thead>
<tr>
<th>Order n</th>
<th>Coordinate ξ_i</th>
<th>Weight w_i</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.0</td>
<td>2.0</td>
</tr>
<tr>
<td>2</td>
<td>±0.57735 02691 89626</td>
<td>1.0</td>
</tr>
<tr>
<td>3</td>
<td>±0.77459 66692 41483 0.0</td>
<td>0.55555 55555 55556 0.88888 88888 88889</td>
</tr>
<tr>
<td>4</td>
<td>±0.86113 63115 94053 ±0.33998 10435 84856</td>
<td>0.34785 48451 37454 0.65214 51548 62546</td>
</tr>
<tr>
<td>5</td>
<td>±0.90617 98459 38664 ±0.53846 93101 05683 0.0</td>
<td>0.23692 68850 56189 0.47862 86704 99366 0.56888 88888 88889</td>
</tr>
</tbody>
</table>

The Gauss method is capable of integrating simple polynomials exactly: A polynomial of order 2n-1 is integrated exactly by \( n \)th order Gauss quadrature. Thus, for instance, a quadratic equation \( (n=2) \), all we need is a 3rd order Gauss quadrature for exact integration. Therefore, for instance, a quadratic equation \( (n=2) \), all we need is a 3rd order Gauss quadrature for exact integration. Thus, for instance, a quadratic equation \( (n=2) \), all we need is a 3rd order Gauss quadrature for exact integration. While \( F(ξ, η) \) involved in element stiffness matrices are generally complicated to know exactly the lowest order needed for exact integration, experience exists with most standard elements. A question arises as to whether it is necessary to perform the integrations exactly or whether there is a limit on the maximum order of integration to use in a given case. Obviously, the lower the order the used the higher the computational efficiency. Recall that in a finite element method, the structure is modeled to be “stiffer” than actual, resulting in smaller displacements than actual. It turns out that the use of higher order Gauss quadrature makes the structure even stiffer. Conversely, the lower order Gauss quadrature renders the structure softer than otherwise. Thus, from this point of view, it is good to use a low order Gauss quadrature. In fact, in subsequent chapters, we will use lower order of integration for pore pressure terms and for certain terms in plate/shell elements. Then the question is a minimum order required. The rule is that the order chosen must at least integrate the length (in 1D) or area (in 2D) or volume (in 3D) exactly. That is, the quantity \( \int |J| \, dξ \) or \( \int |J| \, dξ \, dη \) or \( \int |J| \, dξ \, dη \, dζ \) needs to be integrated exactly. Further discussion is beyond the scope of this book, and the reader is referred to standard texts.

The optimal order of integration for the Q8 element is 3×3 (i.e., \( n=3 \)).

In evaluating the element load vector associated with the body force, the same order of integration as in the evaluation of the element stiffness matrix is used.
In the case of boundary traction, the integrations involved in the element load vectors can mostly be performed exactly. For instance, let us assume that over one of the element boundaries (say, the boundary containing nodes 2, 3 and 6 in Fig. 7), there is a uniform traction normal to the surface. The element load vector to be evaluated is given by Eq. 7b, which is associated with the virtual work terms in Eq. 7a. The integral involved in Eq. 7a can be simplified by recognizing the fact that the variation of the displacement over the boundary depends only on the d.o.f. associated with the nodes on that boundary \((\hat{u}_2, \hat{u}_3, \hat{u}_6)\). Thus, we can define a new modified shape matrix in terms of only \((\hat{u}_2, \hat{u}_3, \hat{u}_6)\) and simplify the integral. Let us do this for a typical element shown in Fig. 11. For simplicity, we will call the nodes as 1, 2 and 3 as shown. Then, Eq. 7a can be modified as:

\[
I_i^3 = \int_{\Gamma_u} \delta \hat{u}_n^T tds = \int_{\Gamma_u} \delta \hat{u}_n f_n ds = \delta \hat{u}_n^T \int_{\Gamma_u} N^T f_n(x)dx
\]

where

\[
\delta \hat{u}_n^T = [\delta \hat{u}_{n1}, \delta \hat{u}_{n1}, \delta \hat{u}_{n1}]
\]

and from Eq. 84c,

\[
N = [N_1, N_2, N_3] = \begin{bmatrix} -\frac{1}{2}(1-\eta)\eta; & \frac{1}{2}(1+\eta)\eta; & (1-\eta^2) \end{bmatrix}
\]

Along the boundary,

\[
x = N_1 x_1 + N_2 x_2 + N_3 x_3 = \begin{bmatrix} -\frac{1}{2}(1-\eta)\eta \end{bmatrix} x_1 + \begin{bmatrix} \frac{1}{2}(1+\eta)\eta \end{bmatrix} x_2 + \begin{bmatrix} (1-\eta^2) \end{bmatrix} x_3
\]

from which, one obtains
\[ \frac{dx}{2} = d\eta \]  \hspace{1cm} (92b)

Then, from Eq. 91c, it follows that the load vector is given by

\[ \begin{bmatrix} P_1 \\ P_2 \\ P_3 \end{bmatrix} = \int_{-1}^{1} \begin{bmatrix} N_1 \\ N_2 \\ N_3 \end{bmatrix} \frac{f_0 \ell}{2} d\eta = \begin{bmatrix} \frac{f_0 \ell}{6} \\ \frac{f_0 \ell}{6} \\ \frac{2f_0 \ell}{3} \end{bmatrix} \]  \hspace{1cm} (93)

Note that the forces in Eq. 93 are equivalent nodal forces perpendicular to the edge shown in Fig. 11. If the edge is not parallel to either \( x \)- or \( y \)-axis, the forces can be resolved in these directions before assembling into the global load vector. The nodal forces corresponding to a shear force traction can be similarly handled.

The details are somewhat involved in the element edge is curved, since both the \( x \)- and \( y \)-components of the displacements and the traction along the edge must now be considered, even when the traction is normal or tangent to the boundary. That only the displacements associated with the nodes on the edge under consideration affects the work done in Eq. 7a is still valid. Thus, we can derive the relevant equations by considering a typical edge such as that shown in Fig. 11b. Eq 7b can be written as:

\[ \begin{bmatrix} P_1 \\ Q_1 \\ P_2 \\ Q_2 \\ P_3 \\ Q_3 \end{bmatrix} = \int_{-1}^{1} \begin{bmatrix} N_1 & 0 \\ 0 & N_1 \\ N_2 & 0 \\ 0 & N_2 \\ N_3 & 0 \\ 0 & N_3 \end{bmatrix} \begin{bmatrix} f_s \cos \theta - f_n \sin \theta \\ f_s \sin \theta + f_n \cos \theta \end{bmatrix} ds \]  \hspace{1cm} (94)

where \( P \) and \( Q \) represent the \( x \)- and \( y \)-components of the nodal load vectors, and \( [N_1 \ N_2 \ N_3] \) are given by Eq. 91e. In performing the integration in Eq. 94, first note that

\[ dx = ds \cos \theta \]  \hspace{1cm} (95a)
\[ dy = ds \sin \theta \]  \hspace{1cm} (95b)

Now noting that \( d\xi = 0 \) on the edge under consideration, from Eq. 87a,

\[ dx = J_{12} d\eta \]  \hspace{1cm} (96a)
\[ dy = J_{23} d\eta \]  \hspace{1cm} (96b)

Now Eq. 94 becomes
$$\begin{bmatrix} P_1 \\ Q_1 \\ P_2 \\ Q_2 \\ P_3 \\ Q_3 \end{bmatrix} = \begin{bmatrix} N_1(f_nJ_{12} - f_nJ_{22}) \\ N_1(f_nJ_{22} + f_nJ_{12}) \\ N_2(f_nJ_{12} - f_nJ_{22}) \\ N_2(f_nJ_{22} + f_nJ_{12}) \\ N_3(f_nJ_{12} - f_nJ_{22}) \\ N_3(f_nJ_{22} + f_nJ_{12}) \end{bmatrix} \int_{-1}^{1} \frac{1}{\eta} d\eta$$

(97)

Fig. 12. Curved Edge of an 8-Noded Isoparametric Plane Element

**Exercise 3.6**

Show that for a linearly varying normal traction over a straight edge, Eq. 93 becomes

$$\begin{bmatrix} P_1 \\ P_2 \\ P_3 \end{bmatrix} = \begin{bmatrix} f_1 \ell \\ \frac{6}{f_2 \ell} \\ \frac{6}{(f_1 + f_2) \ell} \end{bmatrix}$$

(94)

where $f_1$ and $f_2$ are the values of the traction at nodes 1 and 2 respectively.

**Exercise 3.7**

Determine the nodal force vector in Eq. 94 for the element edge shown in Fig. 12, where the element edge is a circular arc. The traction acts normal to the edge.

Now, we will present some details on the calculation of element load vectors for three-dimensional elements. The calculation of element load vectors corresponding to body forces are performed using Eq. 6b using a suitable numerical integration method.
The evaluation of element load vector due to boundary traction involves some further details. The element boundary has a rectangular shape and is a plane, some simplifications are realized, as in the two-dimensional case presented in Eqs. 91a-93. For a 8-noded element boundary as shown in Fig. 13a, for instance, when a uniform traction \( f_0 \) acts normal to the plane, one obtains an equation similar to Eq. 93 for the element load vector normal to the plane as:

\[
\begin{bmatrix}
P_1 \\
P_2 \\
\vdots \\
P_8
\end{bmatrix} = \int \int \begin{bmatrix}
N_1 \\
N_2 \\
\vdots \\
N_8
\end{bmatrix} f_0 \, dx \, dy
\] (95a)

where \( N_1,...N_8 \) are the appropriate shape functions (3D counterparts of those given in Eq. 91e. Noting that \( dx = \frac{\ell}{2} \, d\xi \) and \( dy = \frac{\ell}{2} \, d\eta \), the integral in Eq. 95a can be easily performed. It is left for the reader to show that

\[
[P_1, P_2, ..., P_8] = \begin{bmatrix}
-1/12 & -1/12 & -1/12 & 1/12 & 1/3 & 1/3 & 1/3 & 1/3
\end{bmatrix} f_0 \ell_x \ell_y
\] (95b)

The equivalent nodal forces are shown in Fig. 14 for this case. When the element boundary is not a rectangular as shown in Fig. 13b and/or curved, the formulation is somewhat involved. Let us illustrate the procedure for the case with the traction being
normal to the plane shown in Fig. 13b. \( \mathbf{v}_1 \) and \( \mathbf{v}_2 \) are unit vectors at a given point on the boundary in the \( \xi \)- and \( \eta \)-directions respectively. Let us consider \( \mathbf{v}_1 \) first, which can be expressed as:

\[
\mathbf{v}_1 = i \, dx + j \, dy + k \, dz
\]

where \((dx, dy, dz)\) is small change along \( \mathbf{v}_1 \).

Note that the boundary itself is normal to the 3rd intrinsic coordinate \( \rho \), and \( \rho = \text{constant} \) on the boundary, and as a consequence, \( d\rho = 0 \) on the boundary. As \( \mathbf{v}_1 \) is along the \( \xi \)-axis, \( d\eta = 0 \) along that axis. That is, along the \( \xi \)-axis, \( d\eta = 0 \) and \( d\rho = 0 \).

The 3D Jacobian matrix takes the form:

\[
\begin{vmatrix}
\frac{\partial x}{\partial \xi} & \frac{\partial x}{\partial \eta} & \frac{\partial x}{\partial \rho} \\
\frac{\partial y}{\partial \xi} & \frac{\partial y}{\partial \eta} & \frac{\partial y}{\partial \rho} \\
\frac{\partial z}{\partial \xi} & \frac{\partial z}{\partial \eta} & \frac{\partial z}{\partial \rho}
\end{vmatrix}
= 
\begin{bmatrix}
J_{11} & J_{12} & J_{13} \\
J_{21} & J_{22} & J_{23} \\
J_{31} & J_{32} & J_{33}
\end{bmatrix}
\begin{bmatrix}
d\xi \\
d\eta \\
d\rho
\end{bmatrix}
\tag{97}
\]

from which,

\[
(dx, dy, dz) = (J_{11}d\xi, J_{21}d\xi, J_{31}d\xi) \ .
\]

By a similar logic, the components of \( \mathbf{v}_2 \) are evaluated. Then:

\[
\begin{align*}
\mathbf{v}_1 &= (i \, J_{11} + j \, J_{21} + k \, J_{31})d\xi \quad \text{and} \\
\mathbf{v}_2 &= (i \, J_{12} + j \, J_{22} + k \, J_{32})d\eta
\end{align*}
\tag{98a}
\]

Let \( \mathbf{v}_3 \) be the unit vector perpendicular to the element boundary at the point under consideration. Noting that the magnitude of the cross product \( \mathbf{v}_1 \times \mathbf{v}_2 \) is the area element \( dA \) and the direction coincides with the needed direction \( \mathbf{v}_3 \), one has

\[
\mathbf{v}_3 \, dA = \mathbf{v}_1 \times \mathbf{v}_2 = 
\begin{bmatrix}
J_{21}J_{32} - J_{31}J_{22} \\
-J_{11}J_{32} + J_{31}J_{12} \\
J_{11}J_{22} - J_{21}J_{12}
\end{bmatrix}
\begin{bmatrix}
d\xi \\
d\eta \\
d\rho
\end{bmatrix}
\tag{99}
\]

The element load vector corresponding to a uniform traction normal to the element boundary is now easily evaluated as:

\[
\begin{bmatrix}
P_1 \\
Q_1 \\
R_1
\end{bmatrix}
= 
\int \int \mathbf{v}_3 \, f_n \, dA = 
\begin{bmatrix}
N_1 & 0 & 0 \\
0 & N_1 & 0 \\
0 & 0 & N_1
\end{bmatrix}
\begin{bmatrix}
N_1 & 0 & 0 \\
0 & N_1 & 0 \\
0 & 0 & N_1
\end{bmatrix}
\begin{bmatrix}
J_{21}J_{32} - J_{31}J_{22} \\
-J_{11}J_{32} + J_{31}J_{12} \\
J_{11}J_{22} - J_{21}J_{12}
\end{bmatrix}
\begin{bmatrix}
f_n d\xi d\eta
\end{bmatrix}
\tag{100}
\]

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