Multi-Mechanism Anisotropic Model for Granular Materials

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Abstract

By representing the assembly by a simplified column model, a constitutive theory, referred to as sliding-rolling theory, was recently developed for a two-dimensional assembly of rods subjected to biaxial loading, and then extended to a three-dimensional assembly of spheres subjected to triaxial (equibiaxial) loading. The sliding-rolling theory provides a framework for developing a phenomenological constitutive law for granular materials, which is the objective of the present work. The sliding-rolling theory provides information concerning yield and flow directions during radial and non-radial loading. In addition, the theory provides information on the role of fabric anisotropy on the stress-strain behavior and critical state shear strength. In the present paper, a multi-axial phenomenological model is developed within the sliding-rolling framework by utilizing the concepts of critical state, classical elasto-plasticity and bounding surface. The resulting theory involves two yield surfaces and falls within the definition of the multi-mechanism models. Computational issues concerning the solution uniqueness for stress states at the corner of yield surfaces are addressed. The effect of initial and induced fabric anisotropy on the constitutive behavior is incorporated. It is shown that the model is capable of simulating the effect of anisotropy, and the behavior of loose and dense sands under drained and undrained loading.

Key Words

Constitutive behavior, elasto-plastic materials, granular materials, fabric anisotropy, geologic materials, Multi-mechanism models.
1. Introduction

The stress-strain behavior of granular materials (e.g., sands) is in general quite complex involving volumetric plastic deformation (compaction in some regions and dilation in the others), anisotropy (initial and induced), combined isotropic and kinematic hardening, pressure and density dependencies, liquefaction in some cases under constant volume (undrained) loading resulting in complete loss of strength and stiffness, etc. A large number of phenomenological models have been proposed in the past for granular materials. Most of these models were developed within the framework of the classical elasto-plasticity (Hill, 1950), which employs a set of yield and plastic potential surfaces in the stress space to formulate the constitutive equations. There are, however, other modeling concepts, which avoid the use of surfaces. For example, the hypoplastic framework (Kolymbas, 1991) avoids the use of surfaces, and represents the evolution of the stress rate directly as a function of the strain rate. The other such frameworks are the endochronic framework (Valanis, 1971) and the double-shearing modeling framework (Spencer, 1964, 1982; Mehrabadi and Cowin, 1978).

Some of the existing models have been shown to capture even the most challenging aspects of the behavior; e.g., the liquefaction behavior of sands. While phenomenological approaches offer a powerful means of mathematically capturing its complexity in its entirety, many modeling aspects are arbitrarily assumed in most cases; e.g., flow rule during non-radial deviatoric loading with decreasing deviatoric stress (stress reversal). For example, Anandarajah (1994a, 1994b) proposed a model for granular materials based on a two-surface concept. While the model yields satisfactory results for monotonic
loading behavior (Anandarajah, 1994a) and liquefaction behavior (Anandarajah, 1993, 1994b), due to lack of knowledge concerning the flow rule during stress reversal, the constitutive functions and many of the model parameters lacked physical meaning. Additionally, certain degree of non-smoothness was introduced, which prevented the use of some of most efficient integration algorithms (e.g., the Newton type methods such as the closest point projection algorithm by Simo and Taylor, 1986).

Microscopic approaches have been used in the recent past to model some aspects of the stress-strain behavior, and/or to shed light into the microscopic mechanisms underlying selected macroscopic behavioral features. For example, Rowe (1962) developed a dilatancy rule (a relationship between the ratio of rate of plastic volumetric and deviatoric strains and the stress ratio) by considering the mechanics of a regular array of spheres. The rule says that, during a triaxial compression loading, there is a unique relationship between dilatancy and stress ratio, depending on the interpartcle friction angle. Wan and Guo (1998) extended Rowe’s (1962) theory to consider fabric effects on dilatancy. The dilatancy of real soils depends on the initial void ratio, which is not accounted for in Rowe’s (1962) theory. Also, development of a complete constitutive law requires information on yield and flow directions for other stress probe directions (e.g., radial, stress reversal, etc.), which is not provided by the theory either.

Recently, Anandarajah (2004) developed a microscopic theory for an assembly of two dimensional rods subjected to a biaxial loading and subsequently extended it to represent the behavior of an an assembly of spheres subjected to triaxial loading (Anandarajah,
The plastic strain was considered to be the result of interparticle sliding and rolling deformations. The theory was developed by representing the assembly of particles by load-carrying columns aligned with the directions of the principal stresses. This model is an idealization based on the findings by Cundall and Strack (1979) from discrete element modeling of spheres that an externally applied load to an assembly of spheres is carried by force chains (i.e., load-carrying columns). The columns were connected at “links”, which were represented by an arrangement of four spheres in biaxial analysis and six spheres in triaxial analysis. By considering the statics and kinematics of the link, its behavior was first established. By equating the internal plastic work done to the external plastic work done, the overall stress-strain rate relationships were developed.

The double-shearing model, originally proposed by Mandel (1947), has over the years evolved into a useful constitutive framework for granular materials. In the model, it is assumed that the total plastic deformation is a result of plastic sliding over two active sliding planes. The model was extended for incompressible flow of granular materials by Spencer (1964, 1982) and De Josselin De Jong (1959, 1971). Mehrabadi and Cowin (1978) further extended the double-shearing model to include dilatancy. Nemat-Nasser et al. (1981) then considered work-hardening and pressure sensitivity. Anand (1983) and Balendran and Nemat-Nasser (1993) proposed models for granular materials under plane loading accounting for work-hardening, pressure sensitivity and dilation. Recently, Anand and Gu (2000) further extended the double-shearing model to three-dimensional loading. Zhu et al (2006a, 2006b) extended it to consider the effect of fabric anisotropy in two- and three-dimensional loading conditions. Nemat-Nasser (2000) provided a
comprehensive description of the two-dimensional version of the double-shearing model and Nemat-Nasser and Zhang (2002) presented a three-dimensional version of the double-shearing model. Different hardening rules are employed in different versions of the double-shearing model.

There are some fundamental differences between the sliding-rolling theory and the double-shearing theory. Firstly, the approaches used are different from each other. The sliding-rolling theory has been developed by considering the behavior of an assembly in the principal stress space (of which the triaxial space is a special case). Hence its origin goes back to the work of Rowe (1962). Overall plastic deformation is the result of compaction and expansion along principal stress directions. The dilation, for example, occurs due to expansion being greater than compaction along principal stress (or strain) directions. During the development, the experimental data on triaxial behavior served as target behavior to be simulated (e.g., radial loading, triaxial cyclic loading and liquefaction, loading in triaxial compression and extension, etc.). A good success is obtained when comparing the numerical and theoretical triaxial behaviors.

On the other hand, the double-shearing theory has been developed by considering microscopic behavior on mesoscopic sliding planes, and its origin goes back to the work of Mandel (1947). The overall plastic deformation is the result of dilative simple shear deformations on the mesoscopic sliding planes (Mehrabadi and Cowin, 1978; Nemat-Nasser, 2000). The dilation is considered to result from particles climbing over each other during shear. The theory was largely guided by “shear” type tests (e.g., biaxial shear tests
of Konishi et al., 1982 on rods, hollow cylindrical tests on sands, etc.). The theory has been shown to compare well with plane strain, and hollow-cylindrical test results (Zhu et al., 2006a; Nemat-Nasser and Zhang, 2002). Numerical triaxial simulations are provided in Balendran and Nemat-Nasser (1993), Zhu et al. (2006a) and Nemat-Nasser and Zhang (2002). In a recent paper, Zhu et al. (2006c) compared the results from the double-shearing theory with the results of a hypoplastic theory, verified using triaxial data, and showed good agreements.

Secondly, the motivation behind the development of the sliding-rolling theory is to develop a physical basis and structure for yield and flow directions during radial and non-radial stress probe directions, which can in turn be used to model the complex aspects of the granular material behavior in a more rational manner by a phenomenological model. This approach provides flexibility in modeling complex behavior. On the other hand, the motivation behind the development of double-shearing models has been to develop a micromechanical model.

Several constitutive models proposed in the past considered fabric anisotropy (e.g., Hashiguchi, 1977; Sekiguchi and Ohta, 1977; Ghaboussi and Momen, 1982; Kavadas, 1983; Anandarajah and Dafalias, 1985, 1986; Hashiguchi, 1994; Hashiguchi and Chen, 1998; Chang and Sture, 2006; Hattamleh et al., 2007; Lashkari and Latifi, 2007). Different concepts are used in different models to describe anisotropy. For example, a rotational hardening rule was employed by Hashiguchi (1977), which was originally proposed by Sekiguchi and Ohta (1977). The idea was later adopted for granular
materials by Ghaboussi and Momen (1982), and for cohesive soils by Anandarajah and Dafalias (1986) and Kavvadas (1983), among others, in modeling the initial as well as the induced anisotropy. Hashiguchi (1994) and Hashiguchi and Chen (1998) presented a subloading surface model that considered initial and induced anisotropy by using a rotational hardening rule. Hashiguchi (2001) showed the similarity between the rotational hardening rule used in Hashiguchi and Chen (1998) and the kinematic hardening rule used for metals. Recently, Li and Dafalias (2002) and proposed a model that considers initial anisotropy, but neglects induced anisotropy.

There are many common features among the rotational hardening rules employed in the above models. For example, Anandarajah and Dafalias (1986) controlled the maximum rotation of the surface by controlling the hardening function, introducing what was referred to as the “saturation” phenomenon. Hashiguchi and Chen (1998) limited the rotation by introducing what was referred to as the rotational limit surface. In both the models of Anandarajah and Dafalias (1986) and Hashiguchi and Chen (1998), the rate of change of the tensor modeling the rotation was assumed to be a function of the rate of change of plastic deviatoric strain.

A modified form of the evolution rule used in Anandarajah and Dafalias (1986) is adopted in the present work, which represents the evolution of fabric anisotropy in terms of the plastic strain, but considers the strong dependency on the rate of stress.
Within the double-shearing framework, the models developed by Zhu et al. (2006a, 2006b), Nemat-Nasser (2000) and Nemat-Nasser and Zhang (2002) considered fabric anisotropy. While Zhu et al. (2006a) assumed that the evolution of the fabric tensor depends on the rate of stress, Nemat-Nasser (2000) assumed that it depends on the rate of plastic strain.

The objective of the present paper is to develop a multi-axial phenomenological model for describing the anisotropic stress-strain behavior of granular materials within the framework of the sliding-rolling theory. The resulting theory falls within the definition of the multi-mechanism plasticity models.

**Framework from Sliding-Rolling Theory**

In the notations adopted in the paper, the bold-faced quantities (e.g., $\mathbf{x}$) denote vectors or tensors, and that the tensors are denoted by both the symbolic notations (e.g., $\mathbf{x}$) and the indicial notations (e.g., $x_{ij}$). The theory is limited to small strains and rotations. The total strain rate can be written as $\dot{\varepsilon} = \dot{\varepsilon}^e + \dot{\varepsilon}^p$, where $\dot{\varepsilon}^e$ and $\dot{\varepsilon}^p$ are the rates of elastic and plastic strains. The material is assumed to be transversely isotropic with direction 1 taken as the axis of symmetry (i.e., properties are the same in directions 2 and 3).

In the sliding-rolling theory, the assembly is represented by a set of load-carrying columns aligned with the principal directions of loading (which coincide with the principal directions of material anisotropy). The columns are interconnected at what are referred to as links, consisting of an arrangement of six spheres. By considering the
equilibrium of the links and Coulomb’s friction law at the contacts, it is shown that these links could lose equilibrium only when the stress ratio is changed. When the links do lose equilibrium, they collapse leading to plastic deformation. Hence the plastic deformation by interparticle sliding, according to this model, occurs only during non-radial stress paths. However, it is shown that particle-to-particle rolling can cause plastic deformation even during a radial loading. Considering that particle crushing is negligible at small to moderate levels of stresses, rolling is the only mechanism, according to the theory, that causes plastic deformation at small to moderate levels of stresses. However, it is well known that particle crushing dominates at large stress levels, which is neglected in the sliding-rolling theory. The details may be found in Anandarajah (2004, 2007).

The sliding-rolling theory yields:

$$\dot{\varepsilon}^p = \sum_{k=1}^{3} \lambda_k \mathbf{r}^k$$ \hspace{1cm} (1)

where $$\dot{\lambda}_k = \frac{1}{K_p} \mathbf{n}^k : \dot{\sigma} > 0$$ \hspace{1cm} (2)

$$\dot{\sigma} = \{\dot{p}, \dot{q}\}; \quad \dot{p} = \frac{1}{3}(\dot{\sigma}_1 + 2\dot{\sigma}_3); \quad \dot{q} = \dot{\sigma}_1 - \dot{\sigma}_3$$ \hspace{1cm} (3a)

$$\dot{\varepsilon} = \{\dot{\varepsilon}_v, \dot{\varepsilon}_d\}; \quad \dot{\varepsilon}_v = \dot{\varepsilon}_1 + 2\dot{\varepsilon}_3; \quad \dot{\varepsilon}_d = \frac{2}{3}(\dot{\varepsilon}_1 - \dot{\varepsilon}_3)$$ \hspace{1cm} (3b)

$$\sigma_1, \sigma_2 \text{ and } \sigma_3$$ are the principal stresses, and $$\varepsilon_1, \varepsilon_2 \text{ and } \varepsilon_3$$ are the corresponding principal strains. Note that a triaxial loading is a principal stress loading and is defined as one where $$\sigma_1 \neq \sigma_2 = \sigma_3$$. $$p$$ and $$q$$ are referred to as the mean normal and deviatoric stresses respectively, and $$\varepsilon_v$$ and $$\varepsilon_d$$ are referred to as the volumetric and deviatoric
strains respectively (Schofield and Wroth, 1968). Note that $p \varepsilon_v + q \varepsilon_d = \sigma_1 \varepsilon_1 + 2 \sigma_3 \varepsilon_3$.

Hence $\varepsilon_v$ and $\varepsilon_d$ are the energy conjugates of $p$ and $q$.

The yield directions $n^1$, $n^2$ and $n^3$ and the corresponding flow directions $r^1$, $r^2$ and $r^3$ are schematically shown in Fig. 1, where $n^1$ and $n^2$ are normal to the radial line and $n^3$ is in the direction of the radial line. (Note that the “2” over $n$ in $n^2$ represents Mechanism 2 rather than “square”, etc.) Mechanisms 1 and 2 are associated with sliding and mechanism 3 with rolling. $r^1$ is given as:

$$r^1 = \{r^1_v, r^1_d\}; \quad r^1_v = \sin \beta_v^{\text{max}} - 2 \gamma \cos \beta_v^{\text{max}}; \quad r^1_d = \frac{2}{3} \left( \sin \beta_v^{\text{max}} + \gamma \cos \beta_v^{\text{max}} \right)$$

(4a)

$$\beta_v^{\text{max}} = \tan^{-1}(A_v S_v) + \phi_\mu; \quad A_v = \frac{n_1}{n_3}; \quad S_v = \frac{\sigma_3}{\sigma_1} \leq 1$$

(4b)

where $S_v$ is stress ratio in the $\sigma_1-\sigma_3$ space, $A_v$ is an index of anisotropy, $n_1$ and $n_3$ are number of particle-to-particle contact normal vectors in directions 1 and 3 respectively, $\beta_v^{\text{max}}$ is maximum value of the angle between contact normal and direction 1 at the link level, $\phi_\mu$ is interparticle friction angle, and $\gamma$ is a parameter that varies with the density ($\gamma = \frac{1}{2}$ for loose materials and $\gamma > \frac{1}{2}$ for dense material). The expression for $r^2$ is obtained from Eq. 4 by replacing $\beta_v^{\text{max}}$ with $\beta_v^{\text{min}}$ where $\beta_v^{\text{max}} - \beta_v^{\text{min}} = 2 \phi_\mu$. Direction $r^3$ is a linear combination of $r^1$ and $r^2$, biased towards the positive $q$-axis.

The theory also provides functional forms for $K_p^k$; however, these functional forms are too simplistic for use in the present generalization since the objective of the present
model is to simulate many complex aspects of the granular material behavior. Hence a
more general function is proposed later in the paper. (Actually, hardening rules are
proposed and plastic moduli are derived) When $K_p^k > 0$, among the two sliding
mechanisms (1 & 2), a stress probe that decreases $S_v$ (“forward loading”) will invoke
Mechanism 1 (since $\dot{\lambda}_1 = \frac{1}{K_p^1} n_1 : \dot{\sigma} > 0$) and one that increases $S_v$ (“reverse loading”)
will invoke Mechanism 2 (since $\dot{\lambda}_2 = \frac{1}{K_p^2} n_2 : \dot{\sigma} > 0$). The critical state (CS) failure occurs at constant volume with $\gamma = \frac{1}{2}$. From Eqs. 4a and
4b, at the CS,

$$\beta_v^{\text{max}} = \beta_{\text{CSL}} = 45^0; \quad S_v = S_{\text{CSL}} = \frac{1}{A_{\text{CSL}}} \tan(45 - \phi_\mu); \quad M_{\text{CSL}} = \left[ \frac{q}{p} \right]_{\text{CSL}} = \frac{3(1 - S_{\text{CSL}})}{1 + 2S_{\text{CSL}}}$$

(5)

where $M_{\text{CSL}}$ is slope of the radial line in the $p - q$ space where the CS failure occurs,
known as the critical state line (CSL), and $A_{\text{CSL}}$ is the anisotropic index at the CSL. For
example, for $\phi_\mu = 20^0$ and $A_{\text{CSL}} = 1$ (isotropic), $M_{\text{CSL}} = 0.828$, and for $\phi_\mu = 20^0$ and
$A_{\text{CSL}} = 1.4$ (anisotropic), $M_{\text{CSL}} = 1.2$. Thus, according to the sliding-rolling theory, $M_{\text{CSL}}$
has two contributions: one from $A_{\text{CSL}}$ and another from $\phi_\mu$.

For loose sands, $\gamma$ could start from a value of 0.5 and remain at this value throughout
loading until failure or increase and decrease back to 0.5 at failure. For dense sands, $\gamma$
starts from a value greater than 0.5 and decreases during shear to a value of 0.5 at failure. Correspondingly, for dense sands, the volumetric behavior changes from compaction at the beginning to dilation at a certain radial line. This radial line is referred to as the phase transformation line or PTL (Ishihara, et al., 1975; Habib and Luong, 1978). Hence the parameter $\gamma$ models the dependence of PTL on density (Ishihara, 1993).

The objective of the present paper is to generalize the sliding-rolling theory and develop a constitutive model for granular materials for use in multi-axial loading situations. The framework to be developed falls within the classical definition of the multi-mechanism framework (Hill, 1966; Mandel, 1965; Koiter, 1953; Hodge, 1957; Simo, et al., 1988; Loret, 1990).

2. Proposed Generalized Multi-Mechanism Theory

2.1. The Framework

General

Iwan (1967) and Mroz (1967) proposed the multi-surface concept for developing constitutive models for use in multi-axial loading situations, which was adopted by Prevost (1977) in developing a model for cohesive soils. A two-surface framework, known as the bounding surface framework, was proposed by Dafalias and Papov (1975) and Krieg (1975) for modeling the behavior of metals. Hashiguchi and Ueno (1977) presented a sub-loading surface framework for developing generalized constitutive models. Mroz et al. (1979) later used the two-surface framework for developing a model for cohesive soils. Recently, Manzari and Dafalias (1997) adopted the two-surface
concept in developing a model for granular materials. Prevost (1985) later extended his multi-surface cohesive soil model to granular materials. Due to its simplicity, here we adopt in part (i.e., in the deviatoric plane) a modified form of the two-surface concept in the generalization.

*Yield Surfaces*

First step in the generalization is introducing an yield surface of a finite size at the current stress point (point A in Fig. 1a). This is achieved by defining an elastic region enclosed by two surfaces \( \phi_1 \) and \( \phi_2 \) as shown in Fig. 1b, where \( \phi_1 \) is conical in the meridional plane and circular in the deviatoric plane as shown in Fig. 2b, and \( \phi_2 \) is planar normal to the axis of the cone as shown in Figs. 1b and 2a. Note that open-ended conical yield surfaces have been used by many in the past (e.g., Ghaboussi and Momen, 1982; Poorooshasb and Pietruszczak, 1986; Manzari and Dafalias, 1997). Mechanisms 1 and 2 are then modeled by the normal to the same conical surface \( \phi_1 \) at points B and C respectively (Fig. 1b), and Mechanism 3 is modeled by the normal to the planar surface \( \phi_2 \). Let \( \sigma_{ij} \) and \( \alpha_{ij} \) be the current stress and the center of the cone. The following quantities are defined:

\[
I = \sigma_{kk} ; \quad S_{ij} = \sigma_{ij} - \frac{1}{3} I \delta_{ij} ; \quad J = \left( \frac{1}{2} S_{kl} S_{kl} \right) ; \quad \eta = \frac{J}{I} ; \quad \overline{\eta} = 3 \sqrt[3]{\eta} \quad (6a)
\]

\[
\alpha_{ij}^s = \alpha_{ij} - \frac{1}{3} \alpha_{kk} \delta_{ij} ; \quad \alpha_{ij}^n = \frac{\alpha_{ij}^s}{I} ; \quad \alpha_j^n = \left( \frac{1}{2} \alpha_{kl}^n \alpha_{kl}^n \right)^{1/2} \quad (6b)
\]

\[
\alpha_j = \left( \frac{1}{2} \alpha_{kl}^s \alpha_{kl}^s \right) = I \left( \frac{1}{2} \alpha_{kl}^n \alpha_{kl}^n \right) = I \alpha_j^s \quad (6c)
\]
\[ \eta_{\alpha} = \frac{\alpha_j}{\alpha_{kk}} = \frac{\alpha_j}{I} = \alpha_{jj}^{\eta} ; \quad \eta_{\alpha} = 3 \sqrt{3} \eta_{\alpha} \]  

(6d)

where \( I \) and \( J \) are the trace (1st invariant) and deviatoric stress invariant (2nd invariant) of the stress tensor respectively, \( S_{ij} \) is the deviatoric stress tensor, \( \alpha_{ij}^s \) is the deviatoric part of \( \alpha_{ij} \). \( \alpha_{ij}^{\eta} \) is \( \alpha_{ij}^s \) normalized with \( I \) (tensor of stress ratio). \( \delta_{ij} \) is the Kronecker delta; \( \delta_{ij} = 1 \) for \( i = j \) and \( \delta_{ij} = 0 \) for \( i \neq j \). Hence the \( \delta_{ij} \) represents the direction of the space diagonal or the \( p-q \) axis. \( \eta \) and \( \bar{\eta} \) are stress ratios defined in the \( I-J \) and \( p-q \) spaces respectively at the current stress point. The stress ratios corresponding to the axis of the cone are \( \eta_{\alpha} \) and \( \bar{\eta}_{\alpha} \) in the \( I-J \) and \( p-q \) spaces respectively.

The cases of the forward and reverse loading become special cases of the general 3D case shown in Fig. 2b. For example, the case of forward loading is shown in Fig. 2d. \( \mathbf{n}^1 \) and \( \mathbf{n}^2 \) are associated with the same surface \( \phi_i \) in the general formulation. These will be treated from now on as a single mechanism: Mechanism 1. The third mechanism in Fig. 1 is associated with the second surface \( \phi_2 \) and this will be called Mechanism 2. The notations in Fig. 2 are consistent with these definitions. Hence the three microscopic mechanisms (two sliding and one rolling) are modeled by two phenomenological mechanisms: Mechanism 1 based on \( \phi_1 \) and Mechanism 2 based on \( \phi_2 \).

**Failure Surface and Anisotropy**

The outer surface \( \phi_3 \) seen in Fig. 2b is the failure surface. As will be seen later, the CS failure occurs when \( K_p^1 \) is zero, which occurs on \( \phi_3 \). It may be pointed out that \( \phi_3 \) differs
from what is usually referred to as the bounding surface in the bounding surface models (Dafalias and Papov, 1975) or the normal-yield surface in the subsurface models (Hashiguchi and Ueno, 1977) in that the stress point is allowed to go outside of \( \phi_3 \). The flow direction and yield direction are defined only on the basis of \( \phi_1 \) (and not on the basis of \( \phi_3 \) as is done in the bounding surface models). \( \phi_3 \) defines a surface of zero plastic modulus, and when the plastic modulus is negative, the stress point lies outside of \( \phi_3 \). Hence \( \phi_3 \) and \( \phi_1 \) are permitted to intersect. At the CS, however, the current stress point lies both on \( \phi_1 \) and \( \phi_3 \), with the plastic modulus being zero; hence, \( \phi_3 \) is properly called the failure surface.

As in Anandarajah and Dafalias (1986), a second order tensor \( \delta_{ij}^a \) is used to represent the axis of \( \phi_3 \) as shown in Fig. 2a, where

\[
\delta_{ij}^a \delta_{ij}^a = 3 \quad (7a)
\]

\[
\delta_{ij}^a = \delta_{ij} \quad \text{for isotropic state, and} \quad (7b)
\]

\[
\delta_{ij}^a \neq \delta_{ij} \quad \text{for anisotropic state.} \quad (7c)
\]

The center of \( \phi_3 \) (in the deviatoric plane) is then given by

\[
\alpha_i^d = \frac{1}{\delta_{kk}^a} \delta_{ij}^{ad} \quad (7d)
\]

where \( \delta_{ij}^{ad} \) is the deviatoric part of \( \delta_{ij}^a \), given by:

\[
\delta_{ij}^{ad} = \delta_{ij}^a - \frac{1}{3} \delta_{kk}^a \delta_{ij} \quad (7e)
\]
\[ \delta^u_j = \left( \frac{1}{2} \delta^a_{ij} \delta^a_{ij} \right)^{1/2} \]  

(7f)

\[ \eta_a = \frac{\delta^a_{kk}}{\delta^u_k} ; \quad \bar{\eta}_a = 3 \sqrt[3]{\eta_a} \]  

(7g)

where \( \delta^a_{ij} \) is the deviatoric invariant of \( \delta^a_{ij} \), and \( \eta_a \) and \( \bar{\eta}_a \) are stress ratios corresponding to \( \delta^a_{ij} \) defined in the \( I - J \) and \( p - q \) spaces respectively (\text{“anisotropy ratios”}). With respect to the principal directions of anisotropy, \( \delta^a_{ij} \) is diagonal. When the material is transversely isotropic, \( \delta^a_{11} \neq \delta^a_{22} = \delta^a_{33} \). To quantify anisotropy for this case, define an index as

\[ A = \frac{\delta^a_{11}}{\delta^a_{33}} \]  

(7h)

The index \( A \) will be referred to as the coefficient of anisotropy. \( A = 1 \) for isotropic material state and \( A \neq 1 \) for anisotropic material state. For the general case of material anisotropy, however, \( A \) cannot be used to quantify the degree of anisotropy; \( \eta_a \) or \( \bar{\eta}_a \) (Eq. 7g) may be used.

Our objective is to model the effect of anisotropy based on the results of the sliding-rolling theory, of which, the key aspect is the role of anisotropy on the CS failure. As the effect of \( A_{CSL} \) and \( \phi_\mu \) are coupled in Eq. 5, extension to multi-axial case is not straightforward. Here we use an approximation to uncouple the effects as follows.
Referring to Fig. 3, let us consider a transversely isotropic soil with a coefficient of anisotropy of $A_{\text{CSL}}$. Using Eq. 7a,

$$\delta_{11}^a = A_{\text{CSL}}x_0; \quad \delta_{22}^a = \delta_{33}^a = x_0; \quad x_0 = \sqrt{3} / (2 + A_{\text{CSL}}^2)^{1/2}$$

(8a)

It can be shown that:

$$\delta_{kk}^a = (A_{\text{CSL}} + 2)x_0; \quad \delta_\tau^a = \frac{1}{\sqrt{3}} x_0 (A_{\text{CSL}} - 1); \quad \eta_a = \frac{J_2}{I_1} = \frac{\delta_{kk}^a}{\delta_\tau^a} = \frac{(A_{\text{CSL}} - 1)}{\sqrt{3}(A_{\text{CSL}} + 2)}$$

(8b)

Defining

$$N^{c\delta} = \frac{J_1}{I_1}$$

(8c)

it follows

$$N_{\text{CSL}} = \frac{J_1 + J_2}{I_1} = N^{c\delta} + \frac{(A_{\text{CSL}} - 1)}{\sqrt{3}(A_{\text{CSL}} + 2)}$$

(8d)

$$M_{\text{CSL}} = 3\sqrt{3}N_{\text{CSL}} = M^{c\delta} + \frac{3(A_{\text{CSL}} - 1)}{(A_{\text{CSL}} + 2)}$$

(8e)

where $N_{\text{CSL}}$ and $M_{\text{CSL}}$ are defined in the $I-J$ (invariant) and $p-q$ (triaxial) spaces respectively. Define $M^{c\delta}$ from Eq. 5 for an isotropic assembly as

$$S^{c\delta} = \tan(45 - \phi_p); \quad M^{c\delta} = \frac{q_1}{p_1} = \frac{3(1 - S^{c\delta})}{1 + 2S^{c\delta}} = f_2(\phi_p)$$

(9)

Defining

$$f_1(A_{\text{CSL}}) = \frac{3(A_{\text{CSL}} - 1)}{(A_{\text{CSL}} + 2)}$$

(10a)

it follows
\[ M_{\text{CSL}} = f_1(A_{\text{CSL}}) + f_2(\phi_\mu) \quad (10b) \]

as shown in Fig. 3. The effect of \( A_{\text{CSL}} \) and \( \phi_\mu \) are uncoupled as desired. As illustrated in Table 1 for a typical value of \( \phi_\mu = 25^0 \) and several typical values of \( A_{\text{CSL}} \) (discussed later), Eq. 5 and Eq. 8 provide almost the same values for \( M_{\text{CSL}} \).

Note that in the analysis above, \( A_{\text{CSL}} \) is defined as \( A_{\text{CSL}} = n_1 / n_3 \), where \( n_1 \) and \( n_3 \) are the number of contact normal vectors in direction 1 and 3 respectively. Hence \( \delta''_{ij} \) is defined on the basis of the directional distribution of contact normal vectors, which is related to a fabric tensorial measure as follows.

In micromechanical analyses of granular materials, various measures of fabric anisotropy have been defined and used. For example, the fabric anisotropy based on the statistical distribution of contact normal vectors can be defined as (Oda, 1972; Mehrabadi, et al., 1982; Satake, 1982):

\[
F_{ij} = \int_{\Omega} E(n)n_i n_j d\Omega
\]

where \( n = (n_1, n_2, n_3) \) is unit vector representing the direction of contact normals between two clusters or particles, \( E(n) \) is the statistical distribution function and \( \Omega \) is the volume of the assembly. When the fabric is isotropic, \( F_{ij} \) should be diagonal with equal diagonal members. For a transversely isotropic assembly \( F_{11} \neq F_{22} = F_{33} \). Thus, if one defines an index as
\[ \overline{A} = \frac{F_{11}}{F_{33}} \]  

(12)

then \( \overline{A} = 1 \) for isotropic soil and \( \overline{A} \neq 1 \) for anisotropic soil. For a transversely isotropic soil with the number of contact normal vectors in direction 1 being greater than that in direction 3, \( \overline{A} > 1 \).

Oda (1972) has shown that \( E(n) \) is approximately ellipsoidal for a real soil. Let’s consider an ideal assembly where the contacts normal vectors are either in direction 1, 2 or 3. This is the approximation made in the column model used in the sliding-rolling theory. In this case, Eq. 11 leads to a diagonal matrix with \( F_{11} = n_1 \) and \( F_{22} = F_{33} = n_3 \), where \( n_1 \) and \( n_3 \) are number of contact normal vectors in directions 1 and 3 respectively and \( x \) is an appropriate scale factor. It then follows that \( \overline{A} = n_1 / n_3 \). Hence as far as the direction is concerned, \( \delta_{ij}^a \) may be interpreted as the fabric tensor based on contact normal vectors, \( F_{ij} \) (Eq. 11).

**Flow Rules**

The flow direction \( r_1 \) is obtained directly from the sliding-rolling theory (Eq. 4) as follows. First, consider the case of the loose soil, for which \( \gamma = \frac{1}{2} \). To simulate the CS failure (which is further discussed later), \( r_v^1 \) (Eq. 4a) must be zero when \( \eta = M_{cse} \). This
is achieved as follows. For a given $A_{CSL}$ and $\phi$, compute an equivalent friction angle $\phi_{c,\mu}$ as follows:

$$S_v^c = \frac{1}{A_{CSL}} \tan(45 - \phi_{c,\mu}) = \tan(45 - \phi_{c,\mu}) \quad \Rightarrow \quad \phi_{c,\mu}$$

(13)

To consider the effect of density, $\phi_{c,\mu}$ is modified (as described later) to obtain a value at the PTL, $\phi_{PTL}$. It is required that $\phi_{PTL} \Rightarrow \phi_{c,\mu}$ on the CSL.

For a given value of $\eta$ (Eq. 6d), the ratio between volumetric and deviatoric components of $r^1$ is first computed from the sliding-rolling theory (Eq. 4) as follows:

From $\eta = \frac{3(1 - S_v^a)}{1 + 2S_v^a} \quad \Rightarrow \quad S_v^a = \frac{3 - \eta}{3 + 2\eta}$

(14a)

$$\beta_v^{\max} = \tan^{-1}(S_v^a) + \phi_{PTL}$$

(14b)

$$r_v^1 = \sin \beta_v^{\max} - \cos \beta_v^{\max} ; \quad r_d^1 = \frac{2}{3} \left( \sin \beta_v^{\max} + \cos \beta_v^{\max} \right)$$

(14c)

$$F_i = \frac{r_v^1}{r_d^1}$$

(14d)

Now define

$$r^1 = n^1 + \frac{d \sigma}{I}$$

(15)
The value of \( d \) is calculated as follows. Calculate the forward-loading parameter \( d_f \) by matching \( \frac{r_f}{r_d} \) computed from Eq. 14d with that from Eq. 15. An identical procedure may be used to calculate the reverse-loading parameter \( d_r \). The values of \( d_f \) and \( d_r \) are different from each other. However, to preserve continuity across the \( I - \) axis, it is assumed in the current formulation that \( d = d_f \).

Note that \( \phi_{c\mu} \) computed from Eq. 13 is an interparticle friction angle modified for anisotropy, and not the Mohr-Coulomb friction angle \( \phi_{CSL} \). The latter is related to \( M_{CSL} \) as (Schofield and Wroth, 1968)

\[
M_{CSL} = \frac{6 \sin \phi_{CSL}}{3 - \sin \phi_{CSL}}
\]  

(16)

For example, for \( \phi_{c\mu} = 25^\circ \) and \( A_{CSL} = 1.4 \), \( \phi_{c\mu} = 30.43^\circ \) from Eq. 13, \( M_{CSL} = 1.461 \) (Eqs. 8-10, Table 1), and \( \phi_{CSL} = 36^\circ \) from Eq. 16.

For simplicity, \( r^2 \) is taken to be a modified normal to an elliptical surface as shown in Fig. 2a. The steepness of the ellipse is controlled to ensure solution uniqueness as described subsequently. The ellipse is defined using \( \delta^a_{ij} \) as its major axis. Hence \( r^2 \) is along \( \delta^a_{ij} \) on the \( \delta^a_{ij} \)-axis. As described later, a surface similar to the above ellipse is used to define the plastic modulus associated with Mechanism 1, \( K_p^1 \). To simulate the CS failure, the normal to this surface must be orthogonal to \( \delta_{ij} \) (and not to \( \delta^a_{ij} \)) on the CSL at
the CS failure. To satisfy both of these requirements (i.e., along $\delta^{a}_{ij}$ on the $\delta^{a}_{ij}$-axis and orthogonal to $\delta^{a}_{ij}$ on the CSL), the usual normal to the ellipse must be modified. The following equation satisfies these requirements:

$$\mathbf{r}^2 = \frac{1}{IN \epsilon^{\delta}} \left( \frac{\eta_i}{N^\delta (N^\delta)^2 + \eta_i^2} \right) \mathbf{S}^\delta + \left( N^\delta \right)^2 \left( \frac{\left( (N^\delta)^2 - \eta_i^2 \right)}{(N^\delta)^2 + \eta_i^2} \right) \mathbf{S}^\delta ; \quad \mathbf{S}^\delta = \mathbf{S} - \alpha^\delta \quad (17a)$$

where $\eta_i = \frac{1}{2} \left( \frac{\mathbf{S} \cdot \mathbf{S}^\delta}{\left( \mathbf{S} \cdot \mathbf{S}^\delta \right)^{1/2}} \right)^{1/2} \quad (17b)$

As described later, the relationship between $N^\delta$ and $N^{\epsilon^\delta}$ depends on the density, and the parameters relating the two are chosen with consideration to solution uniqueness. It is easily verified that on the $\delta^{a}_{ij}$-axis, $\mathbf{S}^\delta = 0$ (Figs. 2d and 3), and hence $\mathbf{r}^2$ and $\mathbf{S}^\delta$ have the same direction, and when $N^\delta = \eta_i$, $\mathbf{r}^2$ and $\mathbf{S}^\delta$ have the same direction (and hence $\mathbf{r}^2$ is normal to $\delta^{a}_{ij}$).

**Plastic Constitutive Law and Evolution Rules**

The multi-axial plastic strain rate is given by

$$\dot{\mathbf{e}}^p = \sum_{k=1}^{2} \dot{\lambda}_k \mathbf{r}^k \quad (18a)$$

where $\dot{\lambda}_k = \frac{1}{K^p_k} \mathbf{n}^k : \dot{\mathbf{\sigma}} > 0 \quad (18b)$

The formulation employs three plastic internal variables: $\alpha^\eta_{ij}$, $\delta^{a}_{ij}$ and $p_0$ (Fig. 2), which, respectively introduce kinematic, rotational and isotropic hardening behaviors into the model. The evolution rules used for these variables and the modifications made on the
basis of density (described later) collectively control the hardening and softening behaviors, and the induced anisotropy. The size parameter $k$ associated with $\phi_i$ is assumed to be a constant.

As the primary objective of introducing Mechanism 2 is to simulate the behavior during a radial loading, it is prudent to control the movement of $\alpha''_{ij}$ such that during a radial loading, $\alpha''_{ij}$ will remain unchanged. This restriction avoids unnecessary complications. Furthermore, there is nothing to be gained by allowing $\alpha''_{ij}$ to change during a radial loading. On this basis, the evolution of $\alpha''_{ij}$ is assumed to be invoked only by Mechanism 1 as:

$$\dot{\alpha}'' = \xi_{ij} \dot{\lambda}_i \bar{h}; \quad \bar{h} = \left[ (\bar{S} - S)\eta_s + S(1 - \eta_s) \right] / p; \quad \eta_s = \frac{\delta_s}{\delta_{so}}$$

(19)

where $\bar{S}$, $\delta_s$, and $\delta_{so}$ are defined in Fig. 2. As the stress path for dense material will cross and go outside of $\phi$, $\bar{S} = S$ at the point where it crosses. In this neighborhood, $\bar{S} \approx S$. The use of $\bar{h} = \bar{S} - S$, as is usually done in the two-surface formulation (Mroz et al., 1979), sometimes leads to numerical problems at the crossing point. The expression presented above avoids this. Note that when the inner surface is in contact with the outer surface, $\delta_s = 0$, $\eta_s = 0$ and consequently $\bar{h} = S / p$. Hashiguchi (1988) presented a method for keeping $\phi_i$ inside of $\phi_3$ even when strain softening is involved. In the present model, $\phi_i$ is allowed to cross $\phi_3$. $\xi_{ij}$ is the kinematic hardening function.
Expressions for plastic moduli are obtained from consistency conditions in a subsequent section. The choice for the kinematic hardening function $\xi_1$ controls the specific form of $K_p$.

The evolution of $p_o$ is developed from the CS theory (Schofield and Wroth, 1968) with appropriate modifications, and is discussed in the next section.

Results from biaxial tests on oval-shaped rods suggest that evolution of fabric anisotropy can be related to either the change in stress or change in plastic strain (Oda, et al., 1982; Mehrabadi, et al., 1988). On this basis, Nemat-Nasser (2000) and Nemat-Nasser and Zhang (2002) related the rate of change of fabric tensor to the rate of plastic strain, whereas Zhu et al. (2006a, 2006b) related it to the rate of elastic strain (because the rate of elastic strain is uniquely related to the rate of stress).

The results presented in Fig. 4 were obtained from a recent numerical study by the discrete element method (DEM) of Cundall and Strack (1979). Our own DEM code was used in this study. The tests were conducted on three-dimensional assemblies of spheres by imposing drained triaxial loading under constant mean normal pressure. Fig. 4 presents results for a dense ($e_o, p_{in}) = (0.74, 100 \text{kPa})$ and a loose ($e_o, p_{in}) = (0.84, 467 \text{kPa})$ specimens. A fabric tensor was computed by Eq. 11 and the coefficient of anisotropy $A$ by Eq. 12. It is seen from Fig. 4a that, for the dense specimen, it takes only a very small strain for $\overline{\eta}$ to reach its peak value whereas for the
loose specimen (for which the peak and ultimate strengths are almost the same), it takes a very large strain.

Fig. 4b shows that the variation of $A$ with $\varepsilon_d$ shows a “saturation” phenomenon; i.e., $A$ increases gradually with $\varepsilon_d$ and approaches a limiting value as $\varepsilon_d$ continues to increase. Further support for the trend observed from Fig. 4 may be found in Anandarajah and Kuganenthira (1995). In this study, the variation of electrical conductivities in principal directions of material anisotropy (which coincided with the principal directions of loading) was measured with loading, and the variation of anisotropy was examined by plotting the ratio between principal electrical conductivities with strain.

Fig. 4b shows that $A$ reaches its maximum value in very small strain for the dense specimen and in relatively large strain for the loose specimen. While $A$ is related to $\varepsilon_d$, the relationship is not unique; it depends on the density of the material. Fig. 4c suggests that the relationship between $A$ and $\varpi$ is approximately independent of the density. The behavior seen may be explained as follows: In dense assemblies, the particles are closer to each other, and a very small relative movement (which amounts to very small overall plastic strain) is adequate for a particle to detach contact with its neighbors in one direction and form new contacts in another direction. In loose specimens, large relative movement, and hence large plastic strains are needed for this process to occur. On this basis, the observations made in previous studies are all consistent: $A$ is related to both the plastic strain and stress, but the relationship between $A$ and plastic strain depends on density, whereas that between $A$ and $\varpi$ is almost independent of density. Hence, the rate
of change of anisotropy could still be related to the plastic strain, provided that the density dependence is accounted for. In the interest of adhering to the classical plasticity framework (Hill, 1950) as much as possible, this method is preferred in the present work.

Following Anandarajah and Dafalias (1986), the evolution of $\delta_{ij}^a$ is taken as

$$\dot{\delta}_{ij}^a = \lambda^* (\dot{\lambda}_1 + \dot{\lambda}_2) r_{ij}^{2, ad} ; \quad r_{ij}^{2, ad} = r_{ij}^2 - \frac{1}{3} r_{kk}^{2, ad} \delta_{ij}^a ; \quad r_{kk}^{2, ad} = r_{kk}^2 \delta_{kk}^a$$

(20)

Two important aspects to note here are that (1) both Mechanisms 1 and 2 produce a change in $\delta_{ij}^a$, and (2) the rate of change of $\delta_{ij}^a$ is in the direction orthogonal to $\delta_{ij}^a$.

Thus, when $\phi_3$ has rotated to the point where $\alpha_{ij} = \alpha_{ij}^\delta$ (Fig. 2), and the subsequent loading takes place along a radial line, $\delta_{ij}^a$ no longer changes. However, depending on the limit placed on $A$, $\alpha_{ij}^\delta$ may never become large enough to become equal to $\alpha_{ij}$. The functional form employed for the rotational hardening function $\lambda^*$ dictates the precise nature of the rate of change of $\delta_{ij}^a$, which is presented below after the discussions on the consistency condition. After defining the functional form of plastic moduli, a functional form is chosen for $\lambda^*$ to account for the density effect.

**Determination of the Image Stress**

The surfaces $\phi_1$ and $\phi_3$ are assumed to be circles in the standard octahedral plane; i.e., the plane normal to $\delta_{ij}$. The radii of $\phi_1$ and $\phi_3$ are proportional to the parameters $k$ and $\overline{M}$ respectively; the equations for $\phi_1$ and $\phi_3$ and their relevant properties are given in Appendix 1. Following Mroz et al. (1979), using the proportionality rule
\[
\frac{\bar{S} - \alpha^\delta}{\bar{M}^\delta} = \frac{S - \alpha^S}{k^*} \quad \rightarrow \quad \bar{S} = \alpha^\delta + \frac{M^\delta}{k^*} (S - \alpha^S)
\] (21)

The relationship between \( k \) and \( k^* \) is given in Appendix 1.

2.2. The Critical State (CS)

The pressure and density dependences are two distinguishing features of the stress-strain behavior of soils. When a granular material is subjected to a deviatoric loading starting from a hydrostatic state, it always compacts at the beginning. Whether or not the subsequent behavior is one of compaction or dilation depends on the initial state, defined by the combination of \( p \) and \( e \) where \( p \) is the mean normal pressure and \( e \) is the void ratio. Also, regardless of the initial state, sheared adequately, the material reaches a unique state known as the critical state, first discovered by Casagrande (Taylor, 1948) and subsequently formalized by Schofield and Wroth (1968) for cohesive soils. The critical state was referred to as the steady state by Castro et al. (Castro, 1975; Castro and Poulos, 1975) for granular materials. The CS is defined by a radial line in the \( p - q \) space, known as the critical state line (CSL) in the \( p - q \) space (CSL in \( p - q \) space), and a curve in the \( p - e \) space, known as the critical state line in the \( p - e \) space (CSL in \( p - e \) space).

Ishihara (1993) has conducted a comprehensive investigation and shown, confirming Casagrande’s initial observations, that when the initial state is below the CSL in \( p - e \) space (“dense”), the behavior is dilative and when it is above (“loose”), the behavior of one of compaction. In addition, when the sand is dense, during drained shearing, the stress point first goes above the CSL in \( p - q \) space and then falls back to the CSL,
exhibiting a “peak” on the stress-strain relation, whereas loose sand does not exhibit a peak.

Been and Jefferies (1985) proposed the use of a parameter called the state parameter to quantitatively describe the initial state with respect to the CSL in $p - e$ space. The state parameter is defined as $\psi = e - e_s$ (Fig. 2c). To model the peak behavior of sands, Wood et al. (1994) proposed the use of a peak stress ratio, expressed as a function of the state parameter in such a way that at the CS, the peak stress ratio becomes equal to the slope of the critical state line $M_{CSL}$. The idea was later followed by Manzari and Dafalias (1997) and Li and Dafalias (2000) in developing a generalized model for sands.

Ishihara (1993) found that a different index, known as the state index, correlates better with the stress-strain behavior of sands, both in the low-to-medium and high range of stresses; this index is adopted here. Referring to Fig. 2c, the state index is defined as

$$I_s = \frac{e_u - e}{e_u - e_s}$$

(22)

The upper reference line represents the loosest possible states of the material. Ishihara (1993) uses the void ratio associated with the quasi-steady state in Eq. 22 for $e_s$, although he suggests that either the steady-state (CSL) or the quasi-steady state void ratio could be used. In addition, he uses the straight line $e = e_0$ for the upper reference line in the low range of stresses for Toyura sand. For simplicity, a single upper reference line, taken to be parallel to the CSL, is adopted here. Also, $e_s$ is defined on the steady-state line, rather than the quasi-steady state line.
It may be easily verified that starting from a value of zero on the upper reference line, \( I_s \) increases gradually as the material state moves below the upper reference line and assumes a value of 1 on the CSL in the \( p - e \) space.

In the \( p - e \) space, while a semi-logarithmic relation fits data better for clays, an exponential relation is found to fit better for granular materials (Pestana and Whittle, 1995; Li and Dafalias, 2000; Jefferies and Been, 2000). The assumed relationships are

\[
\begin{align*}
\text{CSL:} & \quad e = \Gamma_S - \lambda \left( \frac{P}{P_a} \right)^{n_p} \\
\text{Upper reference line:} & \quad e = \Gamma_U - \lambda \left( \frac{P}{P_a} \right)^{n_p} \\
\text{Swelling line:} & \quad e = e^* - \kappa \left( \frac{P}{P_a} \right)^{n_e}
\end{align*}
\]

where \( P_a \) is the atmospheric pressure, and \( \Gamma_S, \Gamma_U, \lambda, \kappa \) and \( n_p \) are model parameters.

The exponent \( n_e \) is assumed to be 0.5, which leads to the well-established relation for the bulk and shear moduli (as shown subsequently).

During an isotropic compression, while a unique virgin compression line may be found in most cases (when cementation, sensitivity, etc., are negligible) in the \( p - e \) space for cohesive soils, the compression line varies with the initial void ratio for granular materials. Consider a hydrostatic loading. Such a loading will only invoke Mechanism 2.
Following Schofield and Wroth (1968), referring to Fig. 5a and using Eqs. 18 and 23, it may be shown

\[ \dot{p}_o = g \dot{\lambda}_2; \quad g = g_t r_v^2 \quad (24a) \]

\[ g_t = \frac{(1 + e_o) p_o}{\lambda n_p \left( \frac{p_o}{p_a} \right)^{n_p} - \kappa n_e \left( \frac{p_o}{p_a} \right)^{n_e}} \quad (24b) \]

where \( r_v^2 \) is the volumetric part of \( r_y^2 \).

Since different virgin compression lines are found during an isotropic compression of granular materials depending on the initial void ratio (Jefferies and Been, 2000), \( p_o \) cannot be uniquely found as the intersection of the swelling line and the virgin compression line. For practical reasons, in the present analysis, \( p_o \) is defined as the intersection of the swelling line and CSL in the \( p - e \) space, as shown in Fig. 5b. For Ersak sand, Jefferies and Been (2000) show the isotropic compression line based on the Cam-clay model (Schofield and Wroth, 1968) plots very near the CSL in relative terms (i.e., between the UR and CSL lines, the isotropic compression line based on the Cam-clay model plots much closer to the CSL than the UR-line. The assumption is also consistent with the use of the state parameter for representing the soil behavior (Been and Jefferies, 1985) for the following reason. When \( n_p = n_e = 0.5 \),

\[ \psi = e - e_c = - \left( \lambda - \kappa \right) \left( \frac{p}{p_a} \right)^{0.5} \left[ OCR^{0.5} - 1 \right] \quad (25) \]
where \( OCR = \frac{p_0}{p} \) is the overconsolidation ratio, with \( p_0 \) defined as in Fig. 5b. OCR is thus another measure of the distance between the current state and the CSL, just as \( \psi \) is a measure of the distance between the current state and the CSL. The relationship between OCR and \( \psi \), however, depends on the value of \( p \).

For a more general loading that invokes both Mechanisms 1 and 2, Eq. 24 is generalized as

\[
\dot{p}_o = \sum_{k=1}^{2} g_k \dot{\lambda}_k ; \quad g_k = g_t r^k_v ; \tag{26}
\]

where \( g_t \) is the same function given in Eq. 24b and \( r^k_v \) is the volumetric part of \( r^{ij}_v \).

### 2.3. Summary of Constitutive Equations

The total strain rate is split into elastic and plastic strain rates as

\[
\dot{\varepsilon} = \dot{\varepsilon}^e + \dot{\varepsilon}^p \tag{27}
\]

The hardening variables used in the model are \( \alpha^q \), \( \delta^a \) and \( p_o \). The plastic stress-strain rate equation and the evolution rules for the hardening variables are

\[
\dot{\varepsilon}^p = \sum_{k=1}^{2} \dot{\lambda}_k r^k ; \quad \dot{\lambda}_k = \frac{1}{K_p} n^k : \dot{\sigma} > 0 \tag{28a}
\]

\[
\dot{\alpha}^q = \xi_1 \dot{\lambda}_1 \dot{\gamma} ; \quad \dot{\gamma} = \left[ (S-S)\eta_s + S(1-\eta_s) \right] / p ; \quad \eta_s = \frac{\delta_s}{\delta_{so}} \tag{28b}
\]

\[
\dot{\delta}^a = \lambda^a (\dot{\lambda}_1 + \dot{\lambda}_2) r^{2,ad} ; \quad r^{2,ad} = r^2 - \frac{1}{3} r^{2,\alpha}_k \delta^a ; \quad r^{2,\alpha}_k = r^{2,\alpha}_k \delta^a_{kl} \tag{28c}
\]

\[
\dot{p}_o = \sum_{k=1}^{2} g_k \dot{\lambda}_k ; \quad g_k = g_t r^k_v \tag{28d}
\]
where \( g_t \) is given by Eq. 24b. The image stress is given by (Eq. 21)

\[
\bar{S} = \alpha^\delta + \frac{M^\delta}{k^s} (S - \alpha^s) \tag{29}
\]

The expression for \( k^s \) is given in Appendix 1. Assuming isotropy, the elastic stress-strain rate equations are

\[
\dot{\sigma}_{ij} = C_{ijkl} \dot{e}^e_{kl}; \quad C_{ijkl} = \left( K - \frac{2}{3} G \right) \delta_{ij} \delta_{kl} + G \left( \delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk} \right) \tag{30}
\]

where \( K \) and \( G \) are the bulk and shear moduli respectively. From Eq. 23c (with \( n_e = 0.5 \)), it can be easily shown

\[
K = 2p_a \frac{1 + e_0}{\kappa} \left[ \frac{p}{p_a} \right]^{0.5} \tag{31a}
\]

and using the standard elasticity relations

\[
G = G_0 p_a \left[ \frac{p}{p_a} \right]^{0.5}; \quad G_0 = 2 \frac{1 + e_0}{\kappa} \frac{3(1 - 2\nu)}{2(1 + \nu)} \tag{31b}
\]

where \( K \), \( G \) and \( \nu \) are the bulk modulus, shear modulus and Poisson’s ratio respectively. The first part of Eq. 31b is the well-established functional form for the shear modulus of granular materials (see, for example, Richart, et al., 1970).

### 2.4. Influence of \( I_s \)

To simulate the influence of the variation in \( \gamma \) depending on the density (recall that from the sliding-rolling theory, \( \gamma = 1/2 \) for loose sand and \( \gamma = 1 \) for dense sands), certain parameters are allowed to vary with density.
Wood et al. (1994) proposed the use of the state parameter to account for the effect of density. In the present work, we follow this approach, but use the state index $I_s$ proposed by Ishihara (1993) instead of the state parameter. Following Li and Dafalias (2000) an exponential variation is used. We impose one further requirement that the parameters such as $N^\delta$ (Eq. 17a) approach their values on the CS when $I_s \to 1$ and $A \to A_{CSL}$ simultaneously. $N^\delta$ and $\overline{N}^\delta$ are defined as:

\begin{align}
N^\delta &= N^{c\delta} f_A \exp\{-c_s(I_s - 1)\} \\
\overline{N}^\delta &= N^{c\delta} f_A \exp\{c_a(I_s - 1)\} \\
f_A &= \exp(\delta_{J,CSL}^\delta - \delta_j^a) \\
\delta_{J,CSL}^\delta &= \frac{A_{CSL} - 1}{\sqrt{2 + A_{CSL}^2}}
\end{align}

(32a) (32b) (32c) (32d)

where $c_s$ and $c_a$ are model parameters, and $\delta_j^a$ is given by Eq. 7f. For loose sand, $I_s \leq 1$, and hence $N^\delta \geq N^{c\delta}$ and $\overline{N}^\delta \leq N^{c\delta}$ as shown in Fig. 6. The trend is opposite for dense sand.

Experimental evidence (e.g., Ishihara, 1993) suggests that the slope of the PTL, $M_{PTL}$, is smaller than $M_{CSL}$ for dense sands. Hence $\phi_{PTL}$ (Eq. 14b) must be allowed to vary with $I_s$. Experiments (Arulmoli et al., 1992) also suggest that some soils are more compactive in triaxial extension than in triaxial compression; hence, the rate of variation with $I_s$ must be different in compression and extension. This is achieved by allowing $\phi_{PTL}$ to vary as a function of the Lode angle $\theta$, which is defined as follows:
\[ J_S = \left( \frac{1}{3} S_{ij} S_{jk} S_{kl} \right)^{1/3}; \quad -\frac{\pi}{6} \leq \theta = \frac{1}{3} \sin^{-1} \left[ \frac{3\sqrt{3}}{2} \left( \frac{J_S}{J} \right)^3 \right] \leq \frac{\pi}{6} \]  

where \( \theta = \pi / 6 \) for triaxial compression and \( \theta = -\pi / 6 \) for triaxial extension. This must, however, be done without introducing a discontinuity as the stress path crosses the \( I \)–axis in the meridional plane (or the origin in the octahedral plane). Based on these considerations, the following equations are proposed:

\[ \phi^c_{\text{PTL}} = \phi_{\text{c,PTL}} f_s \exp\{ - c_{dc} (I_s - 1) \} \]  

(34a)

\[ \phi^e_{\text{PTL}} = \phi_{\text{e,PTL}} f_s \exp\{ - c_{de} (I_s - 1) \} \]  

(34b)

\[ n_\phi = \phi^c_{\text{PTL}} / \phi^e_{\text{PTL}} \]  

(34c)

\[ \phi^\text{ptl}_c = g(n_\phi, \theta) \phi^c_{\text{PTL}}; \quad g(n_\phi, \theta) = \frac{2n_\phi}{1 + n_\phi - (1 - n_\phi) \sin 3\theta}; \quad \bar{\phi}_{\text{PTL}} = \frac{\phi^c_{\text{PTL}} + \phi^e_{\text{PTL}}}{2} \]  

(34d)

\[ \phi_{\text{PTL}} = \phi^\text{ptl}_c + (\bar{\phi}_{\text{PTL}} - \phi^\text{ptl}_c) f_s \]  

(34e)

\[ f_s = \exp\left( - \alpha_i \frac{\eta}{N_{\text{CSL}}} \right) \]  

(34f)

where \( c_{dc} \geq 0 \) and \( c_{de} \geq 0 \) are model parameters. For dense sand, \( \phi^c_{\text{PTL}}, \phi^e_{\text{PTL}} < \phi_{\text{c,PTL}}, \) which according to Eq. 14, leads to \( M_{\text{PTL}} < M_{\text{CSL}} \). Similarly for loose sands, \( \phi^c_{\text{PTL}}, \phi^e_{\text{PTL}} > \phi_{\text{c,PTL}} \) and \( M_{\text{PTL}} > M_{\text{CSL}} \). \( f_s \) is simply a smoothing function and \( \alpha_i \) is chosen such that \( \phi_{\text{PTL}} \) changes from \( \phi^c_{\text{PTL}} \) to \( \phi^e_{\text{PTL}} \) across the \( I \)–axis (where \( \eta = 0 \)) at a rate as high as possible without causing numerical problems; \( \alpha_i = 10.0 \) has been found to be a suitable value. The other smoothing function \( g(n_\phi, \theta) \) (Argyris et al., 1973) was used in many of the previous works (e.g., Anandarajah and Dafalias, 1986; Anandarajah, 1994a, 1994b) and found to work very well.
2.5. Consistency Condition and Plastic Moduli

Noting that \( \phi_1 \) and \( \phi_2 \) are independent of \( \delta' \), the functional forms of \( \phi_1 \) and \( \phi_2 \) are

\[
\phi_1(\sigma, \alpha', p_o) = 0; \quad \phi_2(\sigma, \alpha'', p_o) = 0
\]  

(35)

The gradients of the yield surfaces with respect to the stress are

\[
n^1 = \frac{\partial \phi_1}{\partial \sigma}; \quad n^2 = \frac{\partial \phi_2}{\partial \sigma}
\]  

(36)

Define:

\[
T_1 = \frac{\partial \phi_1}{\partial p_o}; \quad T_2 = \frac{\partial \phi_2}{\partial p_o}
\]  

(37)

\[
P_1 = \frac{\partial \phi_1}{\partial \alpha''}; \quad P_2 = \frac{\partial \phi_2}{\partial \alpha''}
\]  

(38)

The expressions for \( \phi_1, \phi_2, n^1, n^2, T_1, T_2, P_1 \) and \( P_2 \) are given in Appendix 1, where it may be noted that \( T_1 = 0 \).

Applying the consistency condition to the yield surfaces

\[
\dot{\phi} = \frac{\partial \phi}{\partial \sigma} \dot{\sigma} + \frac{\partial \phi}{\partial \alpha'} \dot{\alpha'} + \frac{\partial \phi}{\partial p_o} \dot{p}_o = 0
\]  

(39)

Combining Eqs. 28, 36, 37, 38 and 39:

\[
[A][\dot{\lambda}] = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} \dot{\lambda}_1 \\ \dot{\lambda}_2 \end{bmatrix} = \begin{bmatrix} K_p^1 + \xi_1 P_1 : \bar{h} & 0 \\ T_2 g_1 + \xi_1 P_2 : \bar{h} & K_p^2 + T_2 g_2 \end{bmatrix} \begin{bmatrix} \dot{\lambda}_1 \\ \dot{\lambda}_2 \end{bmatrix} = 0
\]  

(40)

For \( \dot{\lambda} \neq 0 \),

\[
|A| = A_{11}A_{22} - A_{12}A_{21} = 0
\]  

(41)

Eq. 41 is satisfied when the diagonal elements of \( A \) are zero.
\[ A_{11} = 0 \quad \iff \quad K^1_p = -\xi_1 \mathbf{P}_1 : \mathbf{h} \quad (42a) \]
\[ A_{22} = 0 \quad \iff \quad K^2_p = -T_2 g_2 \quad (42b) \]

It may be noted that the expression for \( K^1_p \) when only Mechanism 1 is active is the same as the one given by Eq. 42a, and the expression for \( K^2_p \) when only Mechanism 2 is active is the same as the one given by Eq. 42b. This ensures that, regardless of whether Mechanism 1, Mechanism 2 or both are active, the stress point will not lack behind the movement of the surfaces, ending up inside the surfaces, or move faster, ending up outside the surface.

From Eqs. 24b, 28d and 42b, it is seen that \( K^2_p \) is a function of \( p_0 \) and \( r^2_\gamma \). Note from the equation given in Appendix 1 that \( T_2 < 0 \). The sign of \( K^2_p \) then depends on the sign of \( r^2_\gamma \); \( K^2_p \geq 0 \) for \( \eta_1 \leq \bar{N}^d \) where \( \eta_1 \) is given by Eq. 17b. For \( \eta_p = 0.5 \), \( g_1 \) simplifies to

\[ g_1 = \frac{2(1 + e_0)}{\lambda - \kappa} (p_0 p_a)^{1/2} \quad (43) \]

and hence in this case, \( K^2_p \) is proportional to \( p_0^{1/2} \).

The choice of the kinematic hardening function \( \xi_i \) has the most impact on the model’s ability to simulate most of the salient features of the behavior of granular materials. The stiffness of granular materials decreases with decreasing value of \( p \). To be consistent with the variation of \( K \) (Eq. 31a), \( \xi_i \) is assumed to vary with \( p^{1/2} \). The stiffness also decreases with increasing density. This effect may be modeled either based on \( I_s \) or \( p_0 \);
\( \xi_1 \) is assumed here to be a function of \( I_s \). Most importantly, the influence of \( \eta \) must be represented. It is required that \( K_p^1 \) be zero on the failure surface \( \phi_s \); here \( \xi_1 \) is taken to be proportional to \( \vec{r}_v^2 \) (Fig. 6). Finally, the influence of the distance from the failure surface to the current point, \( \delta_s \) (or \( \eta_s \)), must be modeled. Based on these considerations, the following function is proposed:

\[
\xi_1 = \vec{F}_\xi (I_s, p, \eta_s) \vec{r}_v^2 \quad (44a)
\]

\[
\vec{F}_\xi (I_s, p, \eta_s) = c_{e1} \exp\{c_{e2} (I_s - 1)\} \left( \frac{p}{p_a} \right)^{1/2} \left( \frac{g_{r1}}{g_{r2}} \right) (1 + w_i \eta_s) \quad (44b)
\]

where \( c_{e1} \) and \( c_{e2} \) are model parameters, \( g_{r1} = (r^1 : r^1)^{1/2} \) and \( g_{r2} = (r^2 : r^2)^{1/2} \). \( p_a, g_{r1} \) and \( g_{r2} \) are introduced for dimensional consistency. The exponential term accounts for the density effect. The last term is unity when the stress point reaches the failure surface, where \( \eta_s = 0 \). The primary purpose of this term is to ensure that the behavior is not too soft when the stress point is inside the failure surface. While \( w_i \) could be treated as a model parameter, it is assumed to be a constant here: A value of 5 appears to provide satisfactory results for various cases tried thus far.

It can be shown that \( P : \vec{h} < 0 \). It then follows that the sign of \( K_p^1 \) depends on the sign of \( \vec{r}_v^2 \). For dense sands, \( I_s > 0 \) and hence \( \overline{N}^\delta > N^c \delta \) (Eq. 32b). Since \( \vec{r}_v^2 \) becomes negative when the stress point goes above the \( \eta_i = \overline{N}^\delta \) line (Eqs. 17a and 17b), \( K_p^1 < 0 \) above this line, becoming zero on \( \eta_i = \overline{N}^\delta \).
From Eq. 42a and 44a:

\[
K_p^{1} = -\overline{F}_\xi (\mathbf{P}_i : \overline{h}) \overline{v}_v^2 = F_{\xi} \overline{v}_v^2
\]

(44c)

where \( F_{\xi} = -c_{n1} \exp\{c_{r2}(I_s - 1)\} \left( \frac{p}{p_a} \right)^{1/2} \left( \frac{g_{r1}}{g_{r2}} \right) \left( 1 + w_i \eta_i \right) (\mathbf{P}_i : \overline{h}) \)

(44d)

It was pointed out earlier that the functional form for \( \lambda^* \) must be chosen accounting for density effects. Following an approach similar to that in Anandarajah and Dafalias (1986), the following function is proposed:

\[
\lambda^* = c_0 f_1(\beta) f_2(\eta, \rho) f_3; \quad \rho = \frac{S^\delta \cdot \alpha^\delta}{I^2 N^{clas}}; \quad \beta = \frac{(S - \alpha^\delta) \cdot \alpha^\delta}{I^2 N^{clas} (1 + \eta_d^2)^{1/2}}
\]

(45)

where \( c_0 \) is a constant, \( \rho \) and \( \beta \) are scalar products, and \( S^\delta \) is given by Eq. 17a. The functions \( f_1 \) and \( f_2 \) are introduced for modeling the “saturation” phenomenon and the behavior during stress reversals. \( f_3 \) is a special function whose role is discussed below.

During a triaxial forward loading, when the principal directions of material anisotropy coincide with the principal directions of loading, \( \phi_1 \) and \( \phi_2 \) will be aligned as in Fig. 2d. It is easy to see that both \( \rho \) and \( \beta \) assume positive values. If stress reversal occurs at a certain stage, the stress point ends up at the opposite end of \( \phi_1 \), and when this occurs, \( \beta \) becomes negative. Once the center of \( \phi_1 \) goes past the center of \( \phi_2 \), \( \rho \) also becomes negative.
For simplicity, linear functions are proposed. Let \(-\rho_{\text{CSL}} \leq \rho \leq \rho_{\text{CSL}}\) and \(-\beta_{\text{CSL}} \leq \beta \leq \beta_{\text{CSL}}\). Assuming that during a stress reversal beginning on the CSL, \(\beta\) decreases from unity on the CSL to \(c_1\) on the CSL in extension,

\[
f_1(\beta) = \frac{1+c_1}{2} + \frac{1-c_1}{2} \left( \frac{\beta}{\beta_{\text{CSL}}} \right) ; \quad c_1 \geq 0
\]

(46)

Let \(f_2\) decrease from unity to zero as the anisotropy ratio (Eq. 7g) increases from 0 to \(x\eta_{\text{CSL}}\) on the CSL, as

\[
f_2(\eta_a, \rho) = 1 - \frac{\eta_a}{x\eta_{\text{CSL}}} , \text{ where } x = \frac{1+c_2}{2} + \frac{1-c_2}{2} \left( \frac{\rho}{\rho_{\text{CSL}}} \right)
\]

(47)

Consider a triaxial compression loading (Fig. 2d) with \(x = 1\). \(\eta_a\) increases from 0 to \(\eta_{\text{CSL}}\), and in the process, \(f_2\) decreases from its maximum value of unity to zero, and hence \(\lambda^*\) also decreases from its maximum value to zero. Further change of \(\delta_{ij}^a\) is prevented, simulating the “saturation” phenomenon. If \(\lambda^*\) becomes negative, the direction of \(\delta_{ij}^a\) also must change, decreasing \(\eta_a\) and hence making \(\lambda^*\) positive. Hence, further evolution of \(\delta_{ij}^a\) stops.

Now let’s say that a stress reversal occurs. As the function \(f_2\) is “locked” at zero, \(\eta_a\) cannot change. The role of \(x\) in Eq. 47 is unlocking \(f_2\) from its zero value, so that \(\lambda^*\) takes on a small value during a stress reversal, triggering a change in \(\delta_{ij}^a\). For this to occur, \(x\) must start increasing; it is assumed that \(x\) increases from unity on the CSL (\(\rho = \rho_{\text{CSL}}\)) to a value of \(c_2 > 1\) on the CSL on opposite side (\(\rho = -\rho_{\text{CSL}}\)). Since the role
of \( x \) is merely unlocking \( f_2 \), without loss of generality, it is assumed that \( c_2 > 1 \) is fixed.

From numerical experiments, \( c_2 = 1.01 \) is found to be satisfactory. The rate of change of \( \delta_{ij}^a \) during the stress reversal is controlled by the function \( f_1 \), whose variation is controlled by the model parameter \( c_1 \). \( f_1 \) changes from 1.0 at \( \beta = \beta_{CSL} \) to \( c_1 \) at \( \beta = -\beta_{CSL} \); the smaller the value of \( c_1 \), the smaller the change of \( \delta_{ij}^a \) during stress reversal. In other words, \( c_1 \) controls the rate of decay of anisotropy. (We will present numerical results for the cases with \( c_1 = 1 \) and 0.1 later.)

\( f_3 \) is chosen so that the evolution of \( \delta_{ij}^a \) is as much independent of density as possible.

Consider a loading involving only Mechanism 1. Then from Eq. 28a, 28c, and 44c:

\[
\dot{\delta}^a = \lambda^* \dot{\lambda}_1 \mathbf{r}^{2,ad} = \lambda^* \left[ \frac{1}{K_p} \mathbf{n}_1 : \dot{\mathbf{\sigma}} \right] \mathbf{r}^{2,ad} = c_0 f_1 f_2 f_3 \mathbf{n}_1 : \dot{\mathbf{\sigma}} \left[ \frac{\mathbf{r}^{2,ad}}{F_q F_v} \right] \tag{48}
\]

In Eq. 48, all terms, except for the last term and the as yet undetermined function \( f_3 \), are functions only of the current values of \( \mathbf{\sigma} \) and \( \delta_{ij}^a \) and not of density. It now remains to determine \( f_3 \) such that the multiple of \( f_3 \) and the last term is also independent of density. Considering a triaxial compression loading, after a few manipulations, it can be shown that

\[
\mathbf{r}^{2,ad} = x_1 x_2 \frac{1}{\delta_{kk}^a} \left[ \mathbf{S}^\delta - \frac{1}{3} (\mathbf{S}^\delta : \delta^a) \delta^a \right]; \quad x_1 = \frac{\eta_1}{I(N^{\delta})^3}; \quad x_2 = \frac{f_s^2 (f_s^2 + x^2)}{f_s^2 f_A^2 (f_s^2 + x^2)(f_s^2 - x^2)}
\]

\[
f_s = \exp\{-c_s(I_s - 1)\}; \quad f_A = \exp\{c_{\alpha}(I_s - 1)\}; \quad x = \frac{\eta}{f_A N^{\delta}} \tag{49}
\]
where $S^\delta$ and $\eta_1$ are given by Eq. 17. In Eq. 49, except for the parameter $x_2$, the variables are independent of density. Hence if $f_3 = F_\xi / x_2$, $\lambda'$ becomes independent of density. Also, to further simplify Eq. 48 (so that Eq. 48 can be integrated as done below), the following choice is made for $f_3$.

$$f_3 = \frac{1}{x_2} \frac{F_\xi (\sigma^a_{ik})^2 (N^{<\delta})^3}{p}$$  \hspace{1cm} (50)

Finally, issues concerning the difficulties of determining the initial value of $\delta^{a}_{ij}$ and the parameters governing its evolution are discussed and some methods of addressing them are proposed. Anisotropy makes the behavior relatively stiffer in some directions than in others. Typically, tests where the loading is applied in different directions relative to the principal directions of material anisotropy are required to determine the initial anisotropy and the evolution parameters. Even then, it is sometimes difficult to separate the effect of initial and induced parts. In some applications, as for example in geotechnical engineering, undisturbed specimens are hard to obtain, and large specimens that can be used to cut small specimens at different angles to the principal material directions are almost impossible to obtain. Hence simplified and practical calibration methods are needed for the model to serve a useful purpose in practice.

Based on discrete element studies on idealized spherical assemblies and experiments based on measurements of electrical anisotropy (Anandarajah and Kuganenthira, 1995), it has been found that $A_{CSL}$ may be considered to be a constant for a given assembly and hence regarded as a material parameter. Compared to $c_0$, $A_{CSL}$ is physically meaningful,
takes on values in a narrow range and can be determined easily (as explained in the section on calibration). On this basis, an attempt is made to determine $c_0$ from $A_{CSL}$.

As shown in Fig. 4, $A$ increases approximately from 1 to $A_{CSL}$ as $\eta$ increases from 0 to $M_{CSL}$. Considering a triaxial compression loading, Eq. 48 becomes

$$
\dot{\varepsilon}^a = \frac{c_0 f_1 f_2}{F_\ex} \frac{1}{x_2} \frac{F_\ex}{\eta} \frac{(\delta^c)^2 (N^c)^\delta}{p} \left[ n_x \dot{p} + n_d \dot{q} \right] \frac{\eta_1}{I(N^c)^\delta} x_2 \frac{1}{\delta^c} S^\delta - \frac{1}{3} (S^\delta : \delta^c) \delta^a
$$

$$
\text{51a}
$$

Use is made in simplifying Eq. 51a into Eq. 51b that $\sigma : n = n_x \dot{p} + n_d \dot{q}$, where $n_x$ and $n_d$ are triaxial volumetric and deviatoric parts of $n$, and that $\tilde{\eta} = (n_x \dot{p} + n_d \dot{q}) / \sqrt{3} p$. From Eq. 51, making some approximations, the following integral equation is obtained for the rate of change of $\eta_a$ (the details are presented in Appendix 3):

$$
\int_0^{\eta_a^{CSL}} \frac{\dot{\eta}_a}{(a_1 + a_2 \eta_a)(b_1 + b_2 \eta_a)} = \frac{c_0}{27\sqrt{3}} \int_0^{M_{CSL} \eta_a} \tilde{\eta}^2 \tilde{\eta}
$$

$$
\text{52a}
$$

$$
c_0 = \frac{81\sqrt{3}}{b_1 a_2 - b_2 a_1} \ln \left[ \frac{1 + a_2 \eta_a^{CSL}}{1 + b_2 \eta_a^{CSL}} \right] ; \eta_a^{CSL} = \frac{M_{CSL} - 1}{\sqrt{3}(M_{CSL} + 2)}
$$

$$
\text{52b}
$$

where the parameters $a_1$, $a_2$, $b_1$ and $b_2$ are defined in Appendix 3. It may now be noted that, as pointed out earlier, the specific form of Eq. 50 is chosen so that Eq. 52a is simple enough for the integration to be performed.
The discrete element analysis conducted on spheres indicate that $A_{C_{SL}} \approx 1.4$. Given $M_{C_{SL}}$ and $\phi$, the value of $A_{C_{SL}}$ can be calculated from Eq. 10b. In addition, the initial value of $\delta_{ij}^a$ need to be calculated. For a transversely isotropic soil, a value for $A_{in}$ is adequate to calculate the initial value of $\delta_{ij}^a$ using Eq. 8a (with $A_{in}$ substituted for $A_{C_{SL}}$ in this equation). The off diagonal terms are zero.

Using the model parameter Set 1 listed in Table 2 (to be discussed below), the behavior of loose ($e_0 = 0.8, p_o = 40$ kPa) and dense ($e_0 = 0.65, p_o = 40$ kPa) specimens under triaxial drained loading was simulated starting from isotropic states using the hardening function given by Eq. 45. The results are presented in Fig. 7. By comparing Figs. 4 and 7, it may be seen that the desired results are simulated; the variations of $A$ with $\bar{f}$ for dense and loose specimens are approximately the same.

2.6. Overall Stress-Strain Rate Law and the Continuum Operator

From Eqs. 28a and 30

$$\dot{\sigma} = C \dot{\varepsilon} = C[\dot{\varepsilon} - \varepsilon] = C[\dot{\varepsilon} - \sum \dot{\lambda}_k r^k]$$  \hspace{1cm} (53)

Multiplying through by $n'$

$$n' \dot{\sigma} = n' C[\dot{\varepsilon} - \sum \dot{\lambda}_k r^k] = n' C \dot{\varepsilon} - \sum \dot{\lambda}_k n' C r^k$$  \hspace{1cm} (54a)

$$\dot{\lambda}_k K'_p = n' C \dot{\varepsilon} - \sum \dot{\lambda}_k n' C r^k$$  \hspace{1cm} (54b)

$$[B]\dot{[\lambda]} = \begin{bmatrix} K'_p + n' C r^1 & n' C r^2 \\ n^2 C r^1 & K'_p + n^2 C r^2 \end{bmatrix} \begin{bmatrix} \dot{\lambda}_1 \\ \dot{\lambda}_2 \end{bmatrix} = \begin{bmatrix} n' C \dot{\varepsilon} \\ n^2 C \dot{\varepsilon} \end{bmatrix} = \begin{bmatrix} O_1 \\ O_2 \end{bmatrix} = [Q]$$  \hspace{1cm} (54c)
\[
\dot{\lambda}_k = \sum B^{-1}_{km} Q_m = \sum B^{-1}_{km} \left[ n^m C \dot{\varepsilon} \right] \tag{54d}
\]

Define \( A_m = n^m C \) and \( E_k = C \dot{r}^k \) \( \tag{55} \)

Substitute Eq. 54d into Eq. 53, and simplify using Eq. 55:

\[
\dot{\sigma} = \bar{D} \dot{\varepsilon} \tag{56a}
\]

\[
\bar{D} = C - \sum_{m=1}^{2} \sum_{k=1}^{2} B^{-1}_{km} \left[ A_m E_k \right] \tag{56b}
\]

Equation 56a is the overall elasto-plastic stress-strain rate relation and Eq. 56b is the continuum tangent operator.

### 3. Uniqueness of Solution

The issue of the uniqueness of solutions based on multi-mechanism constitutive models have been addressed by many including Hill (1966), Mandel (1965), Koiter (1953), Hodge (1957), Simo, et al. (1988) and Loret (1990). As the present model involves non-associated flow rule and plastic volumetric strains, the analysis presented by Loret (1990) is more applicable, and hence is used here to analyze the present model.

Consider a strain-controlled loading. Depending on the direction of the strain rate, either (1) Mechanism 1 only may be invoked \((\dot{\lambda}_1 > 0, \dot{\lambda}_2 \leq 0)\), or (2) Mechanism 2 only may be invoked \((\dot{\lambda}_1 \leq 0, \dot{\lambda}_2 > 0)\), or (3) Mechanisms 1 and 2 may be invoked \((\dot{\lambda}_1 > 0, \dot{\lambda}_2 > 0)\). Equation 54c may be used to investigate solution uniqueness. Consider the first case. From Eq. 54c,
If follows from Eq. 57 that for a strain rate that causes loading (i.e., $Q_1 > 0$), $\dot{\lambda}_1 > 0$ when the denominator $K_p^1 + n^1 Cr^1 > 0$. This requirement permits $K_p^1$ to be negative, as long as the sum $K_p^1 + n^1 Cr^1$ is positive. A similar logic applies to the second case. A systematic analysis for cases involving stress-, strain- or mixed-mode controls may be found in Klisinski et al. (1992).

Now consider the third case (i.e., $\dot{\lambda}_1 > 0, \dot{\lambda}_2 > 0$). Let us say that the current stress point is at a corner of $\phi_1$ and $\phi_2$ such as point A in Fig. 8. As the direction of the rate of strain changes, for $|B| > 0$, there exists three distinct regions (Fig. 8a) within each of which Mechanism 1 (M: 1) alone, Mechanism 2 (M: 2) alone and Mechanisms 1 and 2 (M: 1+2) operate respectively (Loret, 1990). As the value of $|B|$ decreases, the region where M: 1+2 operate shrinks and become non-existent when $|B| = 0$ (Fig. 8b). When $|B|$ further decreases and becomes negative, a region where all three modes (M: 1, M: 2 and M: 1+2) are permissible is created (Fig. 8c). The solution becomes non unique in this region. Loret (1990) concludes that a necessary and sufficient condition for a unique solution to exist is $|B| > 0$, in addition to the conditions $B_{11} > 0$ and $B_{22} > 0$.

To understand the variation of $|B|$, let us further analyze its coefficients. Note that

$$ r_e = Cr = \left\{ K - \frac{2}{3}G \right\} r_{ik} \delta + 2G r = Kr_{ik} \delta + 2G r^d $$

(58)
where \( r^d \) is the deviatoric part of \( r \). Hence, for granular materials (or geologic materials, in general) for which \( r_{kk} \neq 0 \), \( r \) and \( \mathbf{r} \) are not in the same direction (as is the case in metal plasticity, where \( r_{kk} = 0 \)). The scalar \( \mathbf{nCr} \) is

\[
\mathbf{nCr} = \mathbf{n} : \mathbf{r} = \left\{ K - \frac{2}{3} G \right\} r_{kk} n_k + 2G \mathbf{r} : \mathbf{n}
\]

It then follows that the second term in Eq. 59, and hence the magnitude of \( \mathbf{nCr} \) decrease as the angle between \( \mathbf{r} \) and \( \mathbf{n} \) increases, and vise-versa. Consider a simple case where \( \phi_1 \) and \( \phi_2 \) are orthogonal to each other in the sense \( \mathbf{n}^1 : \mathbf{n}^2 = 0 \) as in Fig. 9a. It is easy to verify that as the flow rules become more non-associative (resulting in higher values of \( \theta_1 \) and \( \theta_2 \)), the diagonal terms \( B_{11} \) and \( B_{22} \) decrease, and the off-diagonal terms \( B_{12} \) and \( B_{21} \) increase, resulting in a decrease in \( |\mathbf{B}| \). Hence to ensure uniqueness, the degree of non-associative behavior must be small enough.

To understand the specific problem at hand, let us refer to Fig. 9b, where at failure (i.e., at the CS failure, \( r^1 = r^2 \Rightarrow |\mathbf{B}| = 0 \). Since \( \theta_2 = 0 \) when \( \eta = 0 \), \( |\mathbf{B}| \) assumes a large value on the hydrostatic axis. Hence as the stress ratio increases, \( |\mathbf{B}| \) generally decreases from its large positive value at \( \eta = 0 \) to zero on the CSL. It is necessary to ensure that \( |\mathbf{B}| \) does not become negative before reaching failure. This can be achieved by controlling either or both of \( \theta_1 \) and \( \theta_2 \). To simulate the behavior satisfactorily, we keep \( \theta_1 \) unchanged and examine if it is necessary to control \( \theta_2 \). Referring to Fig. 6, it is easy to see that \( \theta_2 \) decreases as \( N^d \) is increased. Hence \( \theta_2 \) is a function of the parameter \( c_s \) (Eq. 32a).
An important point to note at this point is that the stress point may lie at the corner of $\phi_1$ and $\phi_2$ only for loose sands. This is because for dense sands, $p_0$ is larger than $p$ (Fig. 2) and hence the point lies either on $\phi_1$ and inside of $\phi_2$ or inside of both $\phi_1$ and $\phi_2$ (elastic). Hence the magnitude of $|B|$ is relevant only to loose sands. Considering triaxial compression loading, equations for the coefficients of $|B|$ may be developed as described in Appendix 2. Here the worst conditions ($K_p^1 = K_p^2 = 0$) are assumed. Using these equations and the model parameter Set 1 listed in Table 1, the variation of $|B|$ with $\eta$ was calculated for a loose sand specimen ($e_0 = 0.775, p_0 = 40$ kPa). The results are shown in Fig. 10, which show that there is no need to control in the present theory the value of $c_s$ to keep $|B|$ positive. For the sake of keeping the model parameters to a minimum, it is assumed that $c_s = 0.5$, which keeps the value of $|B|$ as high as possible.

4. Model Parameters

The model parameters to be determined from experimental data, along with the key equations where they are defined, are listed in Table 2. For simplicity, the Poisson’s ratio $\nu$ and the uniqueness parameter $c_s$ (Eq. 23b) are assumed to be fixed ($\nu = 0.3$ and $c_s = 0.5$). Additionally, the size parameter $k$ is required for $\phi_1$, which is taken as small as possible without invoking numerical problems; a value of 0.05 is found to be satisfactory. As pointed out earlier, it is assumed that $n_e = 0.5$. The methods of determining model parameters are discussed below in conjunction with the model behavior governed by the model parameter of interest.
Given the initial void ratio and the initial stresses, the initial values of $p_0$ and $\alpha''$ are calculated as follows. $p_0$ is computed as illustrated in Fig. 5. When $p_0 \leq p$, $p_0$ is assumed to be equal to $p$. $\alpha''$ is computed by running the axis of the cone representing $\phi_0$ through the current stress point. The remaining details are discussed later.

5. Model Behavior and Model Calibration

The schematics shown in Fig. 11 are used to understand the roles of the two mechanisms (Mechanisms 1 and 2) on the behavior under specific loading and initial conditions.

5.1. Isotropic Compression

A feature of the present model that is different from most other models is the provision for simulating not only the behavior during non-radial stress paths, but also during radial stress paths, of which isotropic compression is a special case. The data presented by Jefferies and Been (2000) on Ersak sand is used to investigate the capability of the present model. Fig. 12a shows the data associated with four different loading-unloading tests performed on specimens with four different void ratios (0.845, 0.7005, 0.6465 and 0.601). Also shown in Fig. 12a are points representing the critical states for some specimens whose behaviors are used in the next section for verifying the model’s capability for simulating triaxial behaviors.
To calibrate the model, first the unloading portions are used to obtain a value for \( \kappa \) (Eq. 23c). The points shown in Fig. 12b are the theoretical unloading curves according to Eq. 23c with \( \kappa = 0.0038 \) (Set 3, Table 2).

An approximation to the CSL is obtained by matching experimental points (from the four tests) with Eq. 23a. Also, from the available data, the location of the UR-line is estimated. The optimal values for the parameters are \( \lambda = 0.013 \), \( \Gamma_s = 0.77 \), \( \Gamma_u = 0.92 \) and \( n_p = 0.75 \) (Set 3, Table 2).

Recall that only when the stress point is on or above the CSL in the \( p-e \) space, the current stress point lies on \( \phi_2 \); otherwise, the current stress point lies inside of \( \phi_2 \) and the behavior until the stress point reaches \( \phi_2 \) is elastic. As shown in Fig. 11a, for loose sands, the plastic behavior begins immediately, whereas, as shown in Fig. 11b, for dense sands, the initial behavior is elastic at the start.

Also note that radial loading will only activate Mechanism 2. Among the four tests, only Test 1 begins from a point above the CSL in the \( p-e \) space. Note that the model behavior is elastic for Tests 2 to 4 (because the mean normal pressure to which the experiments were conducted is smaller than \( p_o \) throughout the test) and hence the loading and unloading curves coincide. The theoretical results are compared with experimental data in Fig. 12c; the agreement is reasonable.
5.2. Triaxial Shearing

The qualitative model behavior is first discussed. The model behavior of loose and dense specimens during triaxial drained loading is summarized in Fig. 13. The model parameter Set 2 is used in the simulations.

During a drained test on a loose specimen starting from the hydrostatic state of stress, the stress point lies at the corner of $\phi_1$ and $\phi_2$ at the beginning (e.g., point A, Fig. 11c). For the stress path associated with the standard triaxial test (which has a slope of 3 vertical to 1 horizontal in the $p-q$ space, at some stage during loading, the stress point moves away from $\phi_2$; i.e., the stress point lies on $\phi_1$, but inside of $\phi_2$ (e.g., point B, Fig. 11c). The point continues to lie on $\phi_1$ until failure.

For loose sands, $I_s \leq 1$, and hence $\overline{N}^\delta \leq N^\delta$ and $N^\delta \geq N^\delta$ (Fig. 6). The latter implies that the slope of the PTL is greater than $N^\delta$ and consequently, the behavior remains compactive until failure (Fig. 13a). $p_o$ increases until the CS failure (Eq. 24a). As the soil compacts, the stress point moves closer to the CSL in the $p-e$ space, and as a result, $I_s$ increases (Fig. 13d), $\overline{N}^\delta$ increases, and $N^\delta$ decreases. The inner surface $\phi_1$ approaches the outer failure surface $\phi_3$, but becomes tangent to it only on the CSL, where it experiences a CS failure. $I_s = 1$ at this stage (Fig. 13d). The plastic modulus $K^1_p$ remains positive throughout, and $\phi_1$ and $\phi_3$ never intersect each other. The stress-strain curve is monotonically increasing until failure and exhibits no peak (Fig. 13b). The
simulation is done starting from an isotropic initial state \( A_{\text{in}} = 1 \). The coefficient of anisotropy gradually increases until it reaches its maximum of \( A_{\text{C SL}} = 1.4 \) (Fig. 13c).

During a drained test on a dense specimen, the stress state starts from a point on \( \phi_1 \) and inside of \( \phi_2 \) (e.g., point A, Fig. 11d) and continues to lie on \( \phi_1 \) (e.g., points B and C, Fig. 11d). The volumetric behavior changes from compactive to dilative on the PTL. Since \( c_{\text{dc}} = c_{\text{de}} > 0 \), this occurs at \( \eta_1 < N^{\varepsilon_\delta} \). After an initial increase during the compactive part, \( p_o \) continues to decrease during the dilative part of the behavior (Eq. 24a) until the CS failure. The coefficient of anisotropy gradually increases until it reaches its maximum of \( A_{\text{C SL}} = 1.4 \) (Fig. 13c). The behavior continues to be dilative until failure (Fig. 13a). \( I_s \) decreases from its initial value of greater than unity to unity at the CS failure (Fig. 13d).

When the stress point reaches the failure surface \( \phi_3 \), \( \bar{r}^2_v = 0 \) and hence \( K_p^1 \) becomes zero (Eq. 44c). This represents the peak in the \( q - \varepsilon_1 \) space (Fig. 13b). Since \( c_\alpha > 0 \), the peak occurs at \( \eta_1 > N^{\varepsilon_\delta} \). It must, however, be noted that the actual stress ratio \( \eta \) at which it occurs depends on the value of \( A \) at this point, which may not necessarily have reached its maximum value (i.e., \( A_{\text{C SL}} \)) yet; recall that the analysis presented in Eqs. 51-52 is approximate. When the stress point goes outside the surface (which is allowed in the present formulation), \( \bar{r}^2_v < 0 \) and hence \( K_p^1 < 0 \). The consistency parameter \( \lambda_1 \) may take on positive value if the stress probing is now in the opposite direction; i.e., in the direction such that \( \mathbf{n}^1 : \dot{\sigma} < 0 \) (Eq. 28a).
Considering a triaxial compression loading, the evolution of the surfaces $\phi_1$ and $\phi_3$ at two time steps (say, $n$ and $n+1$) is shown in Fig. 14 for a state before the peak (Fig. 14a) and for the state at the peak (Fig. 14b) on the deviatoric plane. It follows from Eq. 28b that, regardless of the value of $\eta$, the vector $\vec{h}$ is a non-zero vector. Before the peak, $\vec{h}$, $S$ and $\bar{S}$ are all in the same direction at both steps $n$ and $n+1$ (Fig. 14a). At the peak (step $n$ in Fig. 14b), $S = \bar{S}$, and $\vec{h}$, $S$ and $\bar{S}$ are all still in the same direction. Right after the peak (step $n+1$ in Fig. 14b), $\vec{h}$ and $S$ are in the opposite directions. Since the current point is now outside of $\phi_3$, $\vec{r}_v^2 < 0 \Rightarrow \vec{\xi} < 0$, and hence it follows from Eq. 28b that the center of $\phi_1$ moves back towards the origin of $\phi_3$. Consequently, the stress ratio decreases, exhibiting strain softening in the $N^\delta$ space. For this behavior to be plausible, $\phi_3$ must shrink with respect to its center, which is made possible by the decreasing value of $N^\delta$ (Eq. 32b) caused by dilation (Fig. 13a) and the decreasing value of $I_1$ (Fig. 22).

The triaxial behavior during an undrained loading can be understood directly from the corresponding drained behaviors based on the following. During an undrained loading, $\dot{\varepsilon}_v^p + \dot{\varepsilon}_v^e = 0$, which leads to the result that when $\dot{\varepsilon}_v^p < 0$ (compaction), the effective stress path in the $p-q$ space leans to the left of vertical, and vice-versa. Qualitative model behavior obtained using model parameter Set 2 (Table 2) is shown in Fig. 15, where the behavior of three specimens, two of which are loose ($e_m = 0.74, 0.72$) and the third is
dense \( e_{in} = 0.69 \). Except at the very beginning, the stress point moves away from the corner of \( \phi_1 \) and \( \phi_2 \) very quickly for all three specimens (Figs. 11e and 11f). All three specimens experience compaction at the beginning. The slopes of the PTL are, however, different from each other (Eq. 34a). For all three specimens, \( I_s \) approaches unity and \( A \) increases to \( A_{CSL} \) at the CS failure.

The loosest specimen does not dilate until failure, and has the smallest (positive) plastic modulus \( K_p^1 \) (due to the term \( \exp\{c_{12}(I_s-1)\} \) in Eq. 44b and smaller values of \( \bar{r}_v^2 \)). Consequently, the loosest specimen builds up very large pore water pressure (Fig. 15a), and experiences what is known as the “static liquefaction” (i.e., the effective mean normal pressure becomes zero). The specimen fails before fully reaching the CS (Fig. 15d), and before developing the maximum degree of anisotropy (Fig. 15c).

The second loose specimen \( e_{in} = 0.72 \) first reaches a peak in the \( p-q \) space (Fig. 15a), subsequently experiences a decrease in \( q \), reaching what is known as the quasi steady-state (Ishihara, 1993), and then experiences an increase in \( q \) until the CS failure. This occurs if the stress points ends up below the CS line in the \( p-e \) space when it reaches the CS line in the \( p-q \) space. When this occurs, \( I_s > 1 \) on the CS line in the \( p-q \) space, leading to a subsequent dilatant behavior as seen in Fig. 15a.

For the dense specimen \( e_{in} = 0.69 \), \( K_p^1 \) is large enough that a quasi steady-state is never observed; \( q \) continues to increase until the CS failure. If the stress point goes above the

\[ 53 \]
CS line (as is the case here, Fig. 15b), the same mechanism as in the drained behavior causes the stress ratio to decrease until the CS failure.

To examine the model’s ability to capture the plastic behavior during stress reversals, an additional simulation is performed on the dense specimen with the loading continued until point A in Fig. 15a and then reversed. If the behavior during stress reversal is purely elastic (as in classical plasticity models such as the Cam-clay model, Schofield and Wroth, 1968), the unloading stress path will be vertical. According to the present theory, the specimen builds up positive pore water pressure, and consequently, the stress path leans to the left of vertical as seen in Fig. 15a. This is due to Mechanism 2 resulted from the sliding-rolling theory shown in Fig. 1a. The results show that the model, with further development, has the potential for modeling such behaviors as the cyclic liquefaction, where the stress path continues to move to the left until the mean normal pressure becomes zero.

5.3 Model Calibration

From the behaviors discussed above, it is seen that the isotropic compression and triaxial data may be used to calibrate the model. The parameters $\lambda, \kappa, n_p, \Gamma_S$, and $\Gamma_U$ are obtained from isotropic compression test data as illustrated in section 5.1. The value estimated for Ersak sand using the data in Jefferies and Been (2000) is listed in Table 2 (Set 3). $\phi_\mu$, $A_{csl}$ and $M_{csl}$ are related by Eq. 10; if two of these are known, the third can be calculated. For example, the Ersak sand consists of 73% quartz, 22% feldspar and 5% other minerals. The values of $\phi_\mu$ for quartz and feldspar are reported to be in the range
\(20^\circ \leq \phi \leq 30^\circ\) depending on the surface roughness and grain size (Rowe, 1962; Lambe and Whitman, 1979). Been et al. (1991) report that the CS friction angle \(\phi_{CS}\) (Eq. 16) varies from \(13^\circ\) to \(33^\circ\) depending on the initial density of specimens, implying that the CS friction angle is not a constant for a given soil, but is a function of the initial density. For the tests that are considered in this paper, \(M_{CSL}\) varies from 1.1 for the loose specimen to about 1.2 for the dense specimens. The variable CS friction angle is explained in the present theory by the variable degree of anisotropy that the specimen builds up before the test is terminated. For example, the results are reported in Been et al. (1991) only up to a vertical strain of about 24\%, which appears to be inadequate for the true CS to be reached for this sand. On the basis of these considerations, the following values are chosen:

\[\phi = 20^\circ\] and \(A_{CSL} = 1.4\). \(M_{CSL}\) is calculated to be 1.18 (\(\phi_{CSL} \approx 30^\circ\)).

From the results of dense specimens, the peak stress ratio may be used to find \(c_a\); assuming that \(A \approx A_{CSL}\) at the peak, Eqs. 8 and 32b are used. By a similar procedure, the slope of the PTL is used to calculate a value for \(c_{de}\). However, a value must be assumed for \(A\). For example, \(\eta_a\) may be assumed to vary linearly from 0 to \(\eta_a^{\text{max}}\) as \(\eta\) changes from 0 to \(M_{CSL}\). It is assumed that \(c_{de} = c_{de} \cdot c_{r1}\) and \(c_{r2}\) are obtained by trial-and-error. First, by representing \(c_t = c_{r1} \exp\{c_{r2}(I_s - 1)\}\) and assuming \(c_t\) to be a constant for a given specimen, a value for \(c_t\) is evaluated for each of the specimen by matching theoretical and experimental stress-strain curves. The slope and intercept of the \(\ln c_t\)
versus $I_s - 1$ relation are $c_{r2}$ and $\ln c_{n1} I_s$, however, varies during the test; the estimates must be further refined by trial-and-error.

### 5.4. Comparison between Triaxial Data and Theoretical Results

A selected set of triaxial test data presented in Been et al. (1991) on Ersak sand is used for examining the capability of the present model for simulating behavior for non-radial stress paths. Specifically, results of two drained tests (one on loose specimen and another on dense specimen) and one undrained test on a dense specimen are used. The initial void ratios and isotropic confining pressures for these tests are listed in Table 3.

The initial values of $(e, p)$, along with their values at the critical state are plotted in Fig. 16. It may be observed that the initial values of $(e, p)$ for test 5 (T5) plot slightly above the CSL and is thus defined as a “loose” specimen. Similarly, T6 and T8 are defined as “dense” specimens.

The theoretical and experimental results are compared with each other in Fig. 17a and 17b for T5 (drained test on loose specimen), in Fig. 17c and 17d for T6 (drained test on dense specimen) and in Fig. 17e and 17f for T8 (undrained test on dense specimen). Reasonable agreements are obtained in most cases. The theoretical $q - \varepsilon_1$ relationship is predicted to be stiffer than the experimental $q - \varepsilon_1$ relationship for T8. While the model provides a framework, the agreement depends on the specific functions used to represent the variation of stiffness with density and pressure. Another important factor to consider concerns the high degree of sensitivity of the material behavior to density and pressure.
for initial states in the vicinity of the CSL. Consequently, the theoretical equation used to represent the CSL (i.e., Eq. 23a) and the experimental errors in the measurement of the void ratio and pressure have a significant influence on the agreement between the theoretical and experimental results.

5.5. Behavior Associated with Principal Stress Rotations

By definition, the stress-strain response of an anisotropic material depends on the orientation of the principal stresses ($\sigma_1, \sigma_2, \sigma_3$) with respect to the orientation of the principal directions of the material anisotropy. To investigate this, Mould (1983) (also see Mould et al., 1985) used a true triaxial testing device and performed a series of tests involving principal stress rotations. In one series, he first prepared dense sand specimens that were approximately isotropic and subjected them to drained triaxial compression loading ($\sigma_1 > \sigma_2 = \sigma_3$) up to a certain deviatoric load to cause the development of anisotropy. Mould (1983) then unloaded the specimens to the hydrostatic stress state. Then keeping the loading and material anisotropy directions coincident in direction 2, he performed several triaxial compression tests each with the $\sigma_1 - \sigma_3$ axes rotated through an angle $\theta$ from the 1-3 material anisotropy directions. The reader is referred to Mould (1983) for the details on the experimental procedures followed. The variations of the deviatoric stress and the volumetric strain with the deviatoric strain for five different values of $\theta$ ($\theta = 0^0, 30^0, 45^0, 70^0$ and $90^0$) are shown in Fig. 18. Two important aspects are to be noted: (1) The stress-strain response becomes softer as $\theta$ increases, and (2) the volumetric response becomes more compactive as $\theta$ increases. It must be pointed out that the tests were conducted in load-controlled mode, and hence the specimens would have
failed as soon as the peak is reached. Consequently, the test results are available only up to very low strains, and the close examination of the data reveals that the CS of the specimens has not been reached.

As all of the data required to calibrate the present model are not available (e.g., CS line in the $p-e$ space), the observed behavior is only qualitatively simulated here; we use Set 3 listed in Table 2. The loading sequence was exactly simulated. First, we will consider the behavior associated with $\theta = 0^0$ and $90^0$ since the model behavior can easily be understood for these orientations. The results are presented in Fig. 19.

During the initial triaxial compression to a strain of $\varepsilon_1 = 3\%$ that was intended for inducing anisotropy, the center of $\phi_3$ moves along the $\sigma_1$-axis as shown in Fig. 19d. During unloading back to the hydrostatic state of stress, the center of $\phi_3$ first further moves along the $\sigma_1$-axis until the stress point reaches the center of $\phi_3$. At this point, the direction of $r^{2,ad}$ reverses (Fig. 6) and the center of $\phi_3$ moves back towards the origin in the octahedral plane (Eq. 20); the magnitude of the movement depends on the value of $\lambda^*$ (Eq. 45), which in turn depends on the current values of $\eta_a$ (Eq. 47) and $c_1$ (Eq. 46). For $c_1 = 1$, $A$ increases from 1.0 to 1.3549 and then decreases to 1.3535 ($\bar{\eta}_a = 3\sqrt{3}\eta_a = 0.3162$); i.e., anisotropy hardly decays during unloading. In the subsequent test at $\theta = 0^0$, the behavior is straightforward; $A$ increases from 1.3535 to 1.4 on the CSL. Since the three diagonal elements of $\delta^a$ will be different from each other in the test at $\theta = 90^0$, only means of quantitatively analyzing the movement of $\phi_3$ is to
examine the variation of $\eta_a$ instead of $A$; for the test at $\theta = 0^\circ$, $\eta_a$ increases from 0.3162 to its maximum $\eta_a^{CSL} = 0.3529$ (Fig. 19c). The volumetric behavior is first compactive and then dilative (Fig. 19b) as in a standard triaxial compression test.

For the test at $\theta = 90^\circ$, the stress point moves along the $\sigma_3$-axis as shown in Fig. 19d. When the plastic deformation first begins, the center of $\phi_1$ is at the origin as shown in Fig. 19d. The center of $\phi_3$ begins to move in the direction of $r^{2,ad}$, which is shown in the figure. As this occurs, $\eta_a$ begins to decrease at a rate controlled by $c_1$. The simulations are done for two values of $c_1$ and the results are shown in the figure. As intended, the rate of decrease of $\eta_a$ with strain is higher for $c_1 = 1$ than for $c_1 = 0.1$. As the center of $\phi_3$ moves from the $\sigma_1$-axis towards the $\sigma_3$-axis, $\eta_a$ first decreases and then increases. Now comparing the behavior of the tests at $\theta = 0^\circ$ and $\theta = 90^\circ$, two key points may be noted: (1) The stress point is much closer to $\phi_3$ for $\theta = 90^\circ$ than for $\theta = 0^\circ$ at the early stages of loading, and hence $K_p^1$ for $\theta = 90^\circ$ is smaller than for $\theta = 0^\circ$. The stress-strain response for $\theta = 90^\circ$ is accordingly softer than for $\theta = 0^\circ$. (2) The zone of compaction is centered around the origin in the present model. The stress point in both cases of $\theta = 0^\circ$ and $\theta = 90^\circ$ will have to pass through the compaction zone of the same size. However, since $K_p^1$ is smaller for $\theta = 90^\circ$ than for $\theta = 0^\circ$, the specimen experiences greater compaction for $\theta = 90^\circ$ than for $\theta = 0^\circ$. Consequently, the volumetric behavior shown in Fig. 19b is predicted, which is qualitatively in agreement with the behavior seen in Fig. 18; compare
the numerical behaviors up to a strain of 8% with the corresponding experimental behaviors.

In the present model, for the test at $\theta = 90^\circ$, $\phi_3$ eventually moves all the way to the CS, and hence after adequate shearing, the stress-strain curves for both $\theta = 0^\circ$ and $\theta = 90^\circ$ merge together. The strain at which this occurs could be very large depending on the value of $c_i$ as shown in Fig. 19a. At the range of the low strains (say, up to about 10%), the behaviors are very similar to that seen in Fig. 18. The model behavior for all values of $\theta$ is shown in Fig. 20. The behavior is shown only up to a strain of 10% so that it can be compared with Fig. 18; the qualitative agreement is satisfactory.

6. Conclusions

Based on a recently-developed sliding-rolling theory, a generalized, multi-axial rate independent elasto-plastic constitutive relation is developed for granular materials and applied to represent the behavior of sands. The model is a multi-mechanism model, involving two different yield surfaces, plastic moduli and (non-associated) flow rules. The model utilizes the critical state framework. The initial and induced material anisotropy are accounted for. Computational and uniqueness issues arising from the use of the multi-mechanism framework are addressed. It is shown that the model is capable of describing the behavior of loose and dense sands in isotropic compression, and triaxial drained and undrained loading. The model has the capability to simulate the directional variation of the stress-strain response of anisotropic specimens.
7. Appendices

Appendix 1:

Referring to Fig. 2, in the \( p - q \) space, the size of \( \phi_1 \) is assumed to be proportional to the scalar distance from the origin as

\[
k_{pq} = k(p^2 + q^2)^{1/2} = kp(1 + \eta_a^2)^{1/2}
\]  

(60)

where \( \eta_a \) is the stress ratio associated with the axis of the cone in the \( p - q \) space (Eq. 6d) and \( k \) is a size parameter in the \( p - q \) space. The size in the \( I - J \) space is

\[
k_{IJ} = k^* I; \quad k^* = \frac{k}{3\sqrt{3}} (1 + \eta_a^2)^{1/2}
\]  

(61)

Defining

\[
f_{ij} = S_{ij} - I\alpha''_{ij}; \quad f_J = \left(\frac{1}{2} f_{kI} f_{kI}\right)^{1/2}
\]  

(62)

\( \phi_1 \) is expressed as

\[
\phi_1 = f_J - k^* I = 0
\]  

(63)

The relevant gradients of \( \phi_1 \) are

\[
n^1 = \frac{\partial \phi_1}{\partial \sigma} = \frac{f}{2f_J} - \left[ \frac{f : \alpha''}{2f_J} + k^* \right] \delta
\]  

(64a)

\[
P_1 = \frac{\partial \phi_1}{\partial \alpha^n} = -\left(\frac{I}{2f_J}\right) f - \frac{3\sqrt{3}kl}{2(1 + \eta_a^2)^{1/2}} \alpha''
\]  

(64b)

\[
T_1 = \frac{\partial \phi_1}{\partial p_o} = 0
\]  

(64c)

The equations associated with \( \phi_2 \) are
\[
\phi_2 = 9 J \alpha_\eta^\eta + \frac{1}{3} - 27 p_o (\alpha_\eta^\eta)^2 - p_o
\]  \hspace{1cm} (65a)

\[
n^2 = \frac{\partial \phi_2}{\partial \sigma} = \frac{9 \alpha_\eta^\eta}{2 J} S + \frac{1}{3} \delta
\]  \hspace{1cm} (65b)

\[
P_2 = \frac{\partial \phi_2}{\partial \alpha_\eta^\eta} = \left[ \frac{9 J}{2 \alpha_\eta^\eta} - 27 p_o \right] \alpha_\eta^\eta
\]  \hspace{1cm} (65c)

\[
T_2 = \frac{\partial \phi_2}{\partial p_o} = \left[ -27 (\alpha_\eta^\eta)^2 - 1 \right]
\]  \hspace{1cm} (65d)

Appendix 2:

Considering a triaxial compression loading, it can be shown

\[
n^1 = x_{11} S + x_{12} \delta ; \quad x_{11} = \frac{1}{2J} ; \quad x_{12} = -\left[ \alpha_\eta^\eta + k^{*\eta} \right] \hspace{1cm} (66a)
\]

\[
n^2 = x_{21} S + x_{22} \delta ; \quad x_{21} = \frac{9 \alpha_\eta^\eta}{2J} ; \quad x_{22} = \frac{1}{3}
\]  \hspace{1cm} (66b)

\[
r^1 = x_{31} S + x_{32} \delta ; \quad x_{31} = x_{11} + \frac{d}{l} ; \quad x_{32} = x_{12} + \frac{d}{3}
\]  \hspace{1cm} (66c)

\[
r^2 = x_{41} S + x_{42} \delta^{ad} + \left( \frac{x_{42}}{3} \left( \delta_{kk}^\eta + x_{43} \right) \right) \delta ; \quad x_{41} = \frac{y_0 \eta_1}{l} ; \quad x_{42} = \left[ \left( N^\delta \right)^2 - \eta_1^2 - \frac{\eta_1}{\delta_{kk}^\eta} \right] y_0
\]  \hspace{1cm} (66d)

\[
x_{43} = \frac{\eta_1}{3} ; \quad y_0 = \frac{\left( N^\delta \right)^2}{\left( N^\delta \right)^2 + \eta_1^2 [N^\delta]}
\]  \hspace{1cm} (66d)

where \( \eta_1 \) is given by Eq. 17b, and \( \delta^{ad} \) is the deviatoric part of \( \delta^\eta \) (Eq. 7e).

Noting that \( \n \cdot \mathbf{r} = 2G \mathbf{n}^d : \mathbf{r}^d + Kn_{kk} r_{kk} \), where \( \mathbf{n}^d \) and \( \mathbf{r}^d \) are deviatoric parts of \( \mathbf{n} \) and \( \mathbf{r} \) respectively.
Appendix 3:

Let \( \mathbf{u} \) be the unit tensor in the deviatoric plane along \( \mathbf{S} \) (Fig. 2d). Then

\[
\mathbf{S}^{\delta} = \sqrt{2J_1} \mathbf{u} \quad \delta^{ad} = \sqrt{2}\delta_j^a \mathbf{u} \quad \Rightarrow \quad \mathbf{S}^{\delta} : \delta^{ad} = 2J_1\delta_j^a
\]  

(68a)

\[
\mathbf{S}^{\delta} - \frac{1}{3}(\mathbf{S}^{\delta} : \delta^a)\delta^a = \mathbf{S}^{\delta} - \frac{2}{3}J_1\delta_j^a \left\{ \sqrt{2}\delta_j^a \mathbf{u} + \frac{1}{3}\delta_j^a \delta \right\} = \left[ 1 - \frac{2}{3}(\delta_j^a)^2 \right] \mathbf{S}^{\delta} - \frac{2}{9}J_1\delta_j^a\delta_j^a \delta\delta_j^a
\]  

(68b)

For typical values of \( A_{\text{CSL}} \), \( 1 - \frac{2}{3}(\delta_j^a)^2 \approx 1 \). For example, for \( A_{\text{CSL}} = 1.4 \),

\[
\delta_j^{a,\text{CSL}} = \frac{A_{\text{CSL}} - 1}{(2 + A_{\text{CSL}})^{1/2}} = 0.2 \quad 1 - \frac{2}{3}(\delta_j^a)^2 = 1.027 \approx 1
\]

Using Eq. 68b, the deviatoric part of Eq. 51 becomes:

\[
\dot{\delta}^{ad} = c_0f_1f_2\overline{\eta}\eta\delta_j^a\delta_j^a \mathbf{S}^{\delta} \quad \Rightarrow \quad \dot{\delta}_j^a = \frac{1}{\sqrt{3}}c_0f_1f_2\delta_j^a\eta_j^a\overline{\eta}
\]  

(69)

Making the following approximation: \( \eta_i \approx \eta = \frac{\overline{\eta}}{3\sqrt{3}} \), Eq. 69 becomes

\[
\dot{\delta}_j^a = \frac{1}{27\sqrt{3}}c_0f_1f_2\delta_j^a\eta_j^a\overline{\eta}
\]  

(70)

Neglecting the rate of change of volumetric part,
\[
\dot{\eta}_a = \frac{\delta_{kk}^a \dot{\delta}_{kk}^a - \delta_{ij}^a \dot{\delta}_{ij}^a}{(\delta_{kk}^a)^2} \approx \frac{1}{27 \sqrt{3}} c_0 f_1 f_2 \eta^2 \eta \]

(71)

It can be shown that

\[
\rho_{CSL} = 2\eta_{CSL}^a; \quad \beta = \frac{2k \eta_a}{N^{cs}}; \quad \beta_{CSL} = \frac{2k \eta_{CSL}^a}{N^{cs}}
\]

(72)

From Eqs. 46 and 47:

\[
a_1 = \frac{1}{2} (1 + c_1); \quad a_2 = \frac{1}{2} \left( \frac{1 - c_1}{\eta_{CSL}^a} \right)
\]

(73a)

\[
b_1 = 1; \quad b_2 = -\frac{1}{x \eta_{CSL}^a}
\]

(73b)

8. References


Hodge, P. G., 1957. Piecewise linear plasticity. Proc. 9th. ICAM, 8, 65


Table 1: Comparison of Values of $M_c$ Calculated Using Eq. 5 and Eq. 8e.

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<thead>
<tr>
<th>$\phi_\mu (\theta)$</th>
<th>$S^{c\delta}$</th>
<th>$M^{c\delta}$</th>
<th>$A_{CSL}$ (Eq. 8e)</th>
<th>$M_{CSL}$ (Eq. 5)</th>
<th>$S_{CSL}$ (Eq. 5)</th>
<th>$M_{CSL}$ (Eq. 5)</th>
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Table 2: Model Parameters Used in the Paper

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<th>Parameters</th>
<th>$\lambda$</th>
<th>$\kappa$</th>
<th>$\Gamma_U$</th>
<th>$\Gamma_S$</th>
<th>$n_p$</th>
<th>$\phi_\mu$</th>
<th>$c_{dc} = c_{dc}$</th>
<th>$c_\alpha$</th>
<th>$c_{1/1}$</th>
<th>$c_{1/2}$</th>
<th>$c_1$</th>
<th>$A_{CSL}$</th>
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<td>23c</td>
<td>23b</td>
<td>23a</td>
<td>9</td>
<td>13</td>
<td>34a</td>
<td>34b</td>
<td>32b</td>
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## Table 3. Initial State Parameters for Triaxial Data from Been et al. (1991)

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<tr>
<th>Test</th>
<th>Type</th>
<th>Initial Void Ratio</th>
<th>Initial Confining Pressure (kPa)</th>
<th>State</th>
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<tbody>
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<td>Drained</td>
<td>0.776</td>
<td>500</td>
<td>“Loose”</td>
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<td>T6</td>
<td>Drained</td>
<td>0.691</td>
<td>130</td>
<td>“Dense”</td>
</tr>
<tr>
<td>T8</td>
<td>Undrained</td>
<td>0.618</td>
<td>200</td>
<td>“Dense”</td>
</tr>
</tbody>
</table>
Figures

(a) Yield and Flow Directions in $p$-$q$ Space

(b) Generalization to $I$-$J$ Space

Fig. 1. Yield and Flow Directions from the Sliding-Rolling Theory and their Extensions to Multi-Axial Space
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Fig. 9. Flow Rules
Fig. 10. Variation of $|\mathbf{B}|$ with $\eta$
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Fig. 15. Qualitative Triaxial Undrained Model Behavior
Fig. 16. Initial and Critical States for Triaxial Tests
Points: Exp. (Been et al., 1991)  
Lines: Theory

(a) T5 (Loose, Drained)  
(b) T5 (Loose, Drained)  
(c) T6 (Dense, Drained)  
(d) T6 (Dense, Drained)  
(e) T8 (Dense, Undrained)  
(f) T8 (Dense, Undrained)

Fig. 17. Comparison of Experimental and Theoretical Relationships
Fig. 18. Experimental Directional Variation of Stress-Strain Response (Mould, 1983)
Fig. 19. Model Behaviors for 0° and 90° Principal Stress Rotations
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