Finite Difference Homework for Coastal Modeling

The Laplace equation for the velocity potential appropriate to linear water wave theory is given as

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0 \tag{1}$$

The corresponding boundary conditions are that the potential is periodic in time, the velocity is periodic in x, there is no flow through the bottom, and the combined linear free surface boundary condition is $\partial \phi / \partial z - (\omega^2/g)\, \phi = 0$ on $z = 0$.

Assuming that the velocity potential can be expressed as $\phi(x, z, t) = Z(z) \cos(kx - \omega t)$ and substituting into the Laplace equation yields

$$\frac{d^2 Z(z)}{dz^2} - k^2 Z(z) = 0 \tag{2}$$

The separated boundary conditions are:

$$\begin{aligned} \frac{dZ(z)}{dz} &= 0 \text{ on } z = -h & (3) \\ \frac{dZ(z)}{dz} - \frac{\omega^2}{g} Z(z) &= 0 \text{ on } z = 0 & (4) \end{aligned}$$

Solve this two-point boundary value problem, using finite differences. Be sure the accuracy of the equations are consistent. The finite differences approach will give you an eigenvalue problem problem, of which only one eigenvalue is positive. Compare wavenumbers computed this way with those given by the dispersion relationship. Solve for several different values of kh.